DEMOCRATIC AND POPULAR REPUBLIC OF ALGERIA
Ministry de of Higher Education and Scientific Research

Nour Bachir University Centre of El Bayadh
Institute of Sciences
Sciences and Technology Department

Handout for 1st year Engineering Mathematic Cours

TITLE :

LESSONS AND CORRECTED EXERCISES

IN MATHEMATICS – ANALYSIS 1

PRESENTED BY

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2023-2024

Mathematics for Engineers

This course is intended for students in the first year Engineering courses in Sciences and Technology. Its objective is to provide basic mathematical tools for this sector.

Elementary numerical functions as well as equations and inequalities with a real variable correspond to the secondary school and are assumed to be known.

We begin with a reminder of the algebraic notions relating to functions in $\mathbb{R}$. In the next chapter we present the notions of sequences, series and convergences.

Chapter 3 presents basic functions, their properties and their graphical representations which summarize some information about these functions.

Chapter 4 treat the notions of limits and continuity and their applications.

Chapter 5 presents the notion of derivation, approximations (limited developments) and applications, as well as notion of optimizations (minimum and maximum).

Chapter 6 deals with integral calculus and generalized integrals.

We end with a chapter on the ordinary differential equations of order one then linear ordinary differential equations of order two with constant coefficients.

In this document are included many corrected exercises to show the interest and omnipresence of Mathematics in the various sciences (physics, economics, etc.).

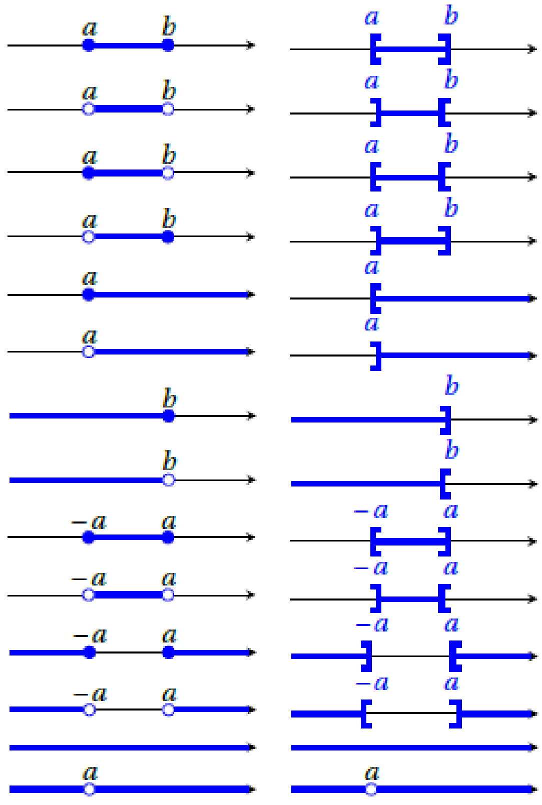
Notations:

Usual sets in mathematics

$\mathbb{N}$: set of natural numbers
$\mathbb{N}^*$: set of natural numbers without zero
$\mathbb{Z}$: set of relative numbers (positives, negatives or zero)
$\mathbb{Z}$: set of relative numbers without zero (positives or negatives)
$\mathbb{Q}$: set of rational numbers ($\frac{p}{q}$ such that $p \in \mathbb{Z}$ and $p \in \mathbb{N}^*$)
$\mathbb{R}$: set of real numbers
$\mathbb{R}^*$: set of natural numbers without zero
$\mathbb{R}$: set of complexe numbers

Intervals

Inequalities	Corresponding set	Graphic representation
$a \le x \le b$	$[a, b]$	
$a < x < b$	$]a, b[$	
$a \le x < b$	$[a, b[$	
$a < x \le b$	$]a, b]$	
$x \ge a$	$[a, +\infty[$	
$x > a$	$]a, +\infty[$	
$x \le b$	$]-\infty, b]$	
$x < b$	$]-\infty, b[$	
$\lvert x\rvert \le a$ avec $a \ge 0$	$[-a, a]$	
$\lvert x\rvert < a$ avec $a \ge 0$	$]-a, a[$	
$\lvert x\rvert \ge a$ avec $a \ge 0$	$]-\infty, -a] \cup [a, +\infty[$	
$\lvert x\rvert > a$ avec $a \ge 0$	$]-\infty, -a[\cup]a, +\infty[$	
$\forall x \in \mathbb{R}$	$]-\infty, +\infty[$	
$x \ne a$	$]-\infty, a[\cup]a, +\infty[= \mathbb{R} \setminus \{a\}$	



Contents

GENERAL INFORMATION ON FUNCTIONS

1st chap. : Properties of the R set

1 EXTREME VALUES

1.1. Upper bound, lower bound

Definition:

If $A \subset \mathbb{R}$ not empty.

- *A real M is a* **majorant** (or **upper bound**) *of A if:*$\forall x \in A \ : \ x \leq M$.
- *A real m is a* **minorant** (or **lower bound**) *of A if:*$\forall x \in A \ : \ x \geq m$.

If an upper bound (resp. a lower bound) of A exists, we say that A is upper bound (resp. lower bound).

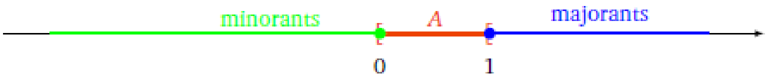
There is not always a majorant or a minorant, in addition we do not have uniqueness.

Examples

1) 3 is an upper bound of]0, 2[;

2)$-7, -\pi, 0$ are minors of $]0, +\infty[$ but there is no upper bound.

3) Let A= [0, 1[.



a). upper bounds of A are exactly the elements of $[1, +\infty[$,

b). lower bounds of A are exactly the elements of $]-\infty, 0]$.

1.2. Infimum, Supremum

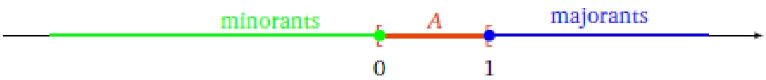
Definition:

If $A \subset \mathbb{R}$ not empty.

- *The* **supremum** *of A is the smallest upper bound. We note :* $\sup A$.
- *The* **infimum** *of A is the largest lower bound. We note :* $\inf A$.

Examples

A = [0, 1[.



1) $\inf A = \min A = 1$. **2)** $\sup A = 1$, but $\max A$ does not exist.

1.3. Maximum, minimum

Definition:

Let be $A \subset \mathbb{R}$ not empty.

- *A real **M** is a **maximum** (or **greatest element**) of A if:*

$$M \in A, \text{ and } \forall x \in A \ : \ x \leq M.$$

If it exists, the greatest element is unique, then we note $M := \max A$.

- *The **minimum** (or **smallest element**) **m** of A, is defined by*

$$m \in A, \text{ and } \forall x \in A \ : \ x \geq m.$$

If it exists, the smallest element is unique, then we note $m := \min A$.

Notice : the maximum or minimum does not always exist.

Examples

1) $\max[a, b] = b \quad , \quad \min[a, b] = a.$

2) The interval]a, b[does not have maximum, nor minimum. (**However, it is bounded**).

3) $\min[0, 1[= 0$ but $\max[0, 1[$ does not exist.

Bounded set:

- *Any subset $A \subset \mathbb{R}$, nonempty and upper bounded admits a **supremum***
- *Any subset $A \subset \mathbb{R}$ nonempty and lower bounded admits a **infimum.***

Notice :

This is the whole point of the **supremum** (resp. **infimum**) compared to the **maximum** (resp. **minimum**) : as soon as a part is bounded it always admits a **supremum** and a **infimum**; which is not the case for the **maximum** and **minimum** as in the example $A = [0, 1[$.

Exercise

Either $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x^2 + 1$ and $A = [-2, 1]$. Determine :

$$1) \ f(A) \quad , \quad 2) \sup_A f \ ou \ \sup_{x \in A} f(x) \quad , \quad 3) \inf_A f \ ou \ \inf_{x \in A} f(x).$$

Correction

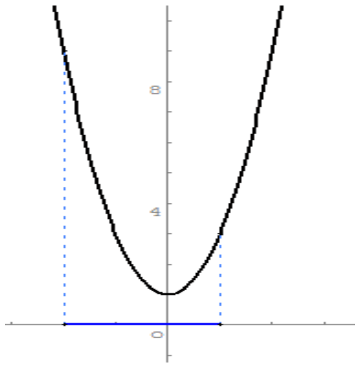
1) $f(A) = [1, 9]$.

2) $\sup_A f = \sup_{x \in A} f(x) := \sup f(A) = 9$.

We notice that the sup is reached in A for $x = -2$; we deduce that $\max_A f = 9$.

3) $\inf_A f = \inf_{x \in A} f(x) := \inf(A) = 1$.

Note that the infimum is reached in A for $x = 0$; we deduce that $\min_A f = 1$.



Question : what can be said about the extrema of f on $\mathbb{R}$?

Exercise

If $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1 - 2x^2$ and $A = [-2, 1]$. Determine :

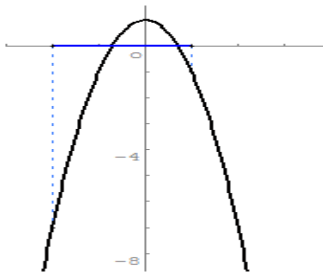
$$1)\ f(A) \quad , \quad 2) \sup_A f \quad , \quad 3) \inf_A f.$$

Correction

1) $f(A) = [-7, 1]$.

2) $\sup_A f(x) = 1 = \max_A f(x)$.

3) $\inf_A f(x) = -7 = \min_A f(x)$.



Question : what can be said about the extrema of f on $\mathbb{R}$?

Exercise

If $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x-1)^2$ and $B = [-1, 1[$: Determine :

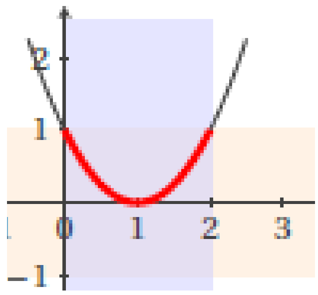
$$1)\ f^{-1}(B) \quad , \quad 2) \sup_B f^{-1} \quad , \quad 3) \inf_B f^{-1}.$$

Correction

1) $f(B) =]0, 2[$.

2) $\sup_B f^{-1} = 2$; $\max_B f^{-1}$ does not exist.

3) $\inf_B f^{-1} = 0$; $\inf_B f^{-1}$ does not exist.



Question : what can be said about the extrema of f on $\mathbb{R}$?

2 FLOOR AND CEILING FUNCTIONS

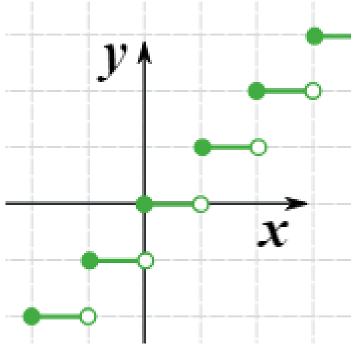
Let $x \in \mathbb{R}$ and $n \in \mathbb{Z}$ is the unique integer such that

$$n \leq x < n + 1.$$

- The **floor function** is defined on $\mathbb{R}$ with values in $\mathbb{Z}$ by

$$floor(x) = n$$

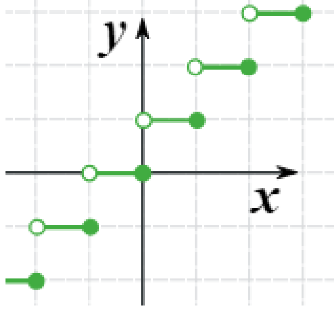
It is usually denoted $\lfloor x \rfloor$. Historically, it has been (and still) called the **integral part** or **integer part** of x, often denoted **[x]**.



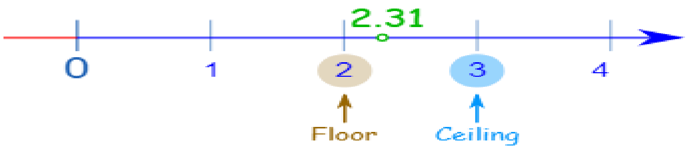
- The **ceiling function** is defined on $\mathbb{R}$ with values in $\mathbb{Z}$ *by*

$$ceil(x) = n + 1$$

It is usually denoted $\lceil x \rceil$.



Example: What is the floor and ceiling of 2,31?



1 LINEAR AND AFFINE FUNCTIONS

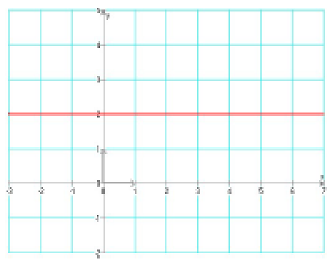
1. *The* **constant function** *is defined on* $\mathbb{R}$ *by*

$$\forall x \in \mathbb{R} \ : \ f(x) = C \ , \ C \in \mathbb{R}.$$

- *It is continuous and indefinitely differentiable, we have:*

$$f'(x) = f''(x) = \ldots = 0$$

- *Its curve is a horizontal line passing through the point (0.C).*



2. *The* **linear function** *is defined on* $\mathbb{R}$ *by*

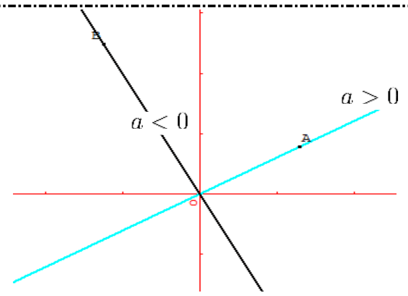
$$\forall x \in \mathbb{R} \; : \; f(x) = ax \, , \; a \in \mathbb{R}.$$

<u>NB :</u> a map f is said to be **linear** if (***by definition***):

$$f(x_1 + x_2) = f(x_1) + f(x_2) \qquad \text{And} \quad f(a\,x) = a\,f(x_1).$$

- *Function* $x \to ax$ *is continuous and indefinitely differentiable, we have:* $(ax)' = a$ *And* $(ax)'' = (ax)''' = \ldots = 0$
- *Its curve is a straight line (D) (**not vertical**) passing through the origin.*
- *The number* $a \in \mathbb{R}$ *is called the* **leading coefficient**. *If the* ***reference*** *is* ***orthonormal****, it is called* **slope**.

- *If* $a > 0$ *the function is increasing and the line is ascending*
- *If* $a < 0$ *the function is increasing and the line is descending*



Knowing a point $A(x_A, y_A) \neq O$ *is enough to determine the line (D)*

- *analytically* [$a = \frac{y_A}{x_A}$] *that is* $y = \frac{y_A}{x_A}\,x$
- *or geometrically* [$(D) = (OA)$].

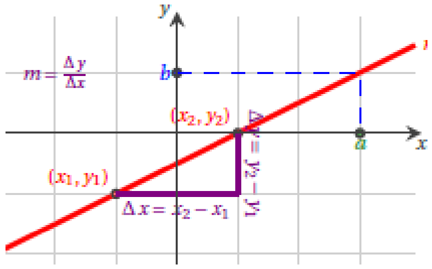
3. *The* **affine function** *is defined on* $\mathbb{R}$ *by*

$$\forall x \in \mathbb{R} \; : \; f(x) = ax + b \; , \; a, b \in \mathbb{R}.$$

- *It is continuous and indefinitely differentiable, we have:*
  $(ax+b)' = a$ *and* $(ax+b)'' = (ax+b)''' = \ldots = 0.$
- *Its curve is a straight line (D) (not vertical).*
- *The number* $a \in \mathbb{R}$ *is called the leading coefficient,*
  the number $b \in \mathbb{R}$ *is called the abscissa at the origin.*
- *I* $a > 0$ *the function is increasing (the line is ascending).*
  If $a < 0$ *the function is decreasing (the line is descending).*

Consider the function: $x \to f(x) = m(x-a)+b$*:*

- C_f *goes through M(a,b).*
- *In a* ***cartesian coordinate system*** $m = \tan\theta$ *where* θ *denotes the angle between the line* C_f *and the axis (x'ox).*



For two lines $C_1 : y = a_1x + b_2$ *and* $C_2 : y = a_2x + b_2$ *we have* $C_1 // C_2 \iff a_1 = a_2$ *and* $C_1 \perp C_2 \iff a_1.a_2 = -1.$

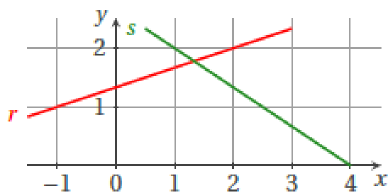
The knowledge of two points $A(x_A, y_A) \neq B(x_B, y_B)$ *is enough to determine the line (D)*

- *analytically:*$y - y_A = \frac{y_B - y_A}{x_B - x_A}(x - x_A)$ *or* $y - y_B = \frac{y_B - y_A}{x_B - x_A}(x - x_B)$*;*
- *geometrically* [$(D) = (AB)$].

$$M(x,y) \in (AB) \iff \overrightarrow{AM} // \overrightarrow{AB} \iff \begin{vmatrix} x - x_A & x_B - x_A \\ y - y_A & y_B - y_A \end{vmatrix} = 0.$$

Exercise

Find the equation of the lines r and s shown below and calculate the coordinates of the point of intersection.



Correction

Graphically we have $r : (AB)$ where $A(-1,1)$ and $B(2,2)$, we deduce :

Analytically $y - y_A = \frac{y_B - y_A}{x_B - x_A}(x - x_A)$ i.e. $y - 1 = \frac{2-1}{2+1}(x+1)$ that is $y = \frac{1}{3}x + \frac{4}{3}$.

<u>Geometrically</u>

$$\begin{aligned} M(x,y)\in(AB) &\iff \overrightarrow{AM}//\overrightarrow{AB} \\ &\iff \begin{vmatrix} x-x_A & x_B-x_A \\ y-y_A & y_B-y_A \end{vmatrix}=0 \\ &\iff \begin{vmatrix} x+1 & 3 \\ y-1 & 1 \end{vmatrix}=x+1-3(y-1)=0 \\ &\iff x-3y+4=0 \end{aligned}$$

Graphically we have $s:(CD)$ where $A(1,2)$ and $B(4,0)$, we deduce :

<u>Analytically</u>: $y-2=\frac{0-2}{4-1}(x-1)$ that is $y=-\frac{2}{3}x+\frac{8}{3}$.

<u>Geometrically</u>

$$\begin{aligned} M(x,y)\in(AB) &\iff \overrightarrow{AM}//\overrightarrow{AB} \\ &\iff \begin{vmatrix} x-x_A & x_B-x_A \\ y-y_A & y_B-y_A \end{vmatrix}=0 \\ &\iff \begin{vmatrix} x-1 & 3 \\ y-2 & -2 \end{vmatrix}=-2(x-1)-3(y-2)=0 \\ &\iff -2x-3y+8=0 \end{aligned}$$

Intersection $M(x,y)$ if it exists, will be the solution of the system

$$\begin{cases} x-3y+4=0 \\ 2x+3y-8=0 \end{cases}, \quad \text{equivalently} \begin{cases} x-3y+4=0 \\ 3x-4=0 \end{cases}, \text{ then} \begin{cases} y=16/9 \\ x=4/3 \end{cases}.$$

<u>Exercise</u>

Let the linear system $(S)\begin{cases} x-ky=1 \\ kx-y=1 \end{cases}$

Geometrically determine the values of $k\in\mathbb{R}$ such that this system has:

1.) an infinity of solutions; 2.) no solution; 3.) one-stop solution.

<u>Correction</u>

Geometrically the system (S) is the intersection of the lines $C_1 \ : \ y=\dfrac{1}{k}x-\dfrac{1}{k}$ and $C_2 \ : \ y=kx-1$, then :

1.) The system has infinitely many solutions if and only if

$$C_1=C_2: \quad \frac{1}{k}=k \ et \ -\frac{1}{k}=-1 \ \textbf{i.e.} \quad k=1.$$

2.) The system has no solution if and only if

$$C_1//C_2: \quad \frac{1}{k}=k \ et \ -\frac{1}{k}\neq-1 \ \textbf{i.e.} \quad k=-1.$$

3.) The system has a unique solution. $\dfrac{1}{k}\neq k$ **i.e.** $k\in\mathbb{R}\setminus\{-1,+1\}$.

1.1. Sign of the 1st degree polynomial

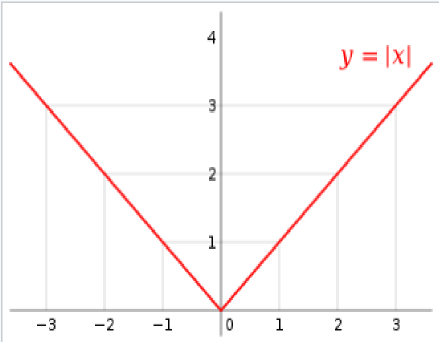
x	$-\infty$	$-\frac{b}{a}$	$+\infty$
$ax+b$	sign of $(-a)$		sign of $(+a)$

2 ABSOLUTE VALUE

The **absolute value function** *is defined on* $\mathbb{R}$ *by*

$$\forall x \in \mathbb{R} \ : \ f(x) = |x| := \begin{cases} -x & \text{si } x \leq 0 \\ +x & \text{si } x \geq 0 \end{cases}$$

- *Function* $x \to |x|$ *is continuous.*
  It is not differentiable at x=0
- *Its curve is a broken line at (0,0).*



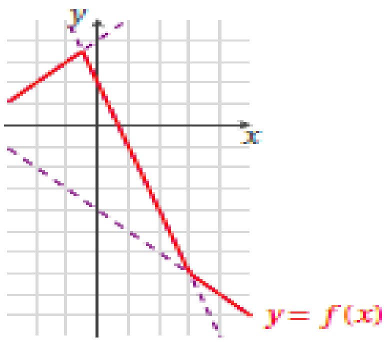
The graph of the absolute value function for real numbers

Exercise

Let $f : x \to |x-3| - |2x+1|$ *defined on* $\mathbb{R}$. Simplify the expression of $f(x)$ then plot a curve representative of the function f.

.***Correction***

x	$-\infty$	$-\frac{1}{2}$ $\quad 3$	$-\infty$
$\|x-3\|$	$-x+3$	$-x+3$	$x-3$
$-\|2x+1\|$	$2x+1$	$-2x-1$	$-2x-1$
$f(x)$	$x+4$	$-3x+2$	$-x-4$



$$f(x) = \begin{cases} x+4 & \text{if } -\infty < x \leq -1/2 \\ -3x+2 & \text{if } -1/2 < x \leq 3 \\ -x-4 & \text{if } 3 < x < +\infty \end{cases}$$

Exercise

Find all solutions in $\mathbb{R}$ of the following inequalities :

1) $x\,|x| < 1$ **2)** $|x+2| < 1 + |x-1|$

.***Correction***

1)

	$-\infty$	0 $\qquad$ $+\infty$
$x\left\|x\right\| < 1$	$-x^2 < 1$ i.e. $x^2 > -1$	$x^2 < 1$ i.e. $-1 < x < +1$
Solutions	$]-\infty, 0]$	$[0, 1[$

We deduce : $x\left|x\right| < 1 \iff x \in]-\infty, 1[$.

2)

x	$-\infty$	-2	1 $\qquad$ $+\infty$
$\left\|x+2\right\|$	$-x-2$	$x+2$	$x+2$
$\left\|x-1\right\|$	$-x+1$	$-x+1$	$x-1$
inequality	$-x-2 < 1-x+1$ i.e. $-2 < 2$	$x+2 < 1-x+1$ i.e. $2x < 0$	$2 < 0$
Solutions	$]-\infty, -2]$	$[-2, 0[$	No solutions

We deduce : $\left|x+2\right| < 1 + \left|x-1\right| \iff x \in]-\infty, 0[$.

2nd chap. : Numerical sequences

1 Definitions

Definition (Suites):

A sequence is an application $u:\mathbb{N}\to\mathbb{R}$.

For $n\in\mathbb{N}$, we note $u(n)$ by u_n and we call it **nth term**, **term of order n** *or* **general term** *of the sequence.*

Notice :

In practice, there are essentially two methods for defining a sequence:

1) we define$(x_n)_{n\in\mathbb{N}}$ directly as a function of n, for example

$$\forall n\in\mathbb{N}^*: u_n=\frac{1}{n^2}.$$

whose first terms are $u_1=1, u_2=\frac{1}{4}, u_3=\frac{1}{9}, u_4=\frac{1}{16}\cdots$

2) or we define the sequence by **recurrence**, for examples :

a) the arithmetic sequence of ratio $(r\in\mathbb{R})$ and first term u_0 :

$$\forall n\in\mathbb{N}: u_n=u_0+n\,r$$

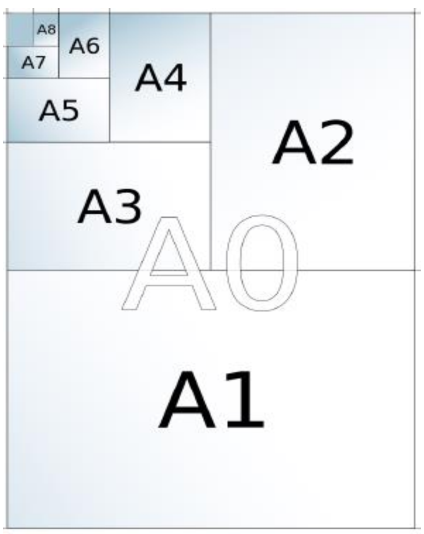
b) the geometric sequence of ratio $(q\in\mathbb{R})$ and first term u_0:

$$\forall n\in\mathbb{N}: u_n=u_0\,q^n$$

Example : ***Size of a sheet of paper***

The format of a rectangular sheet of paper is the couple formed by its width and its length. This format varies according to the use of the sheet, the period and the geographical area. For common uses, especially in office automation, the A4 format is now very widespread.

The format $A_n$is designed so that the proportions of the sheet are maintained when it is folded or cut in half along its length, thus avoiding loss in bookmaking by folding, assembly, enlargement and reduction by the factor of two .



The number n in A_n indicates the number of times the basic format A_0 was divided into two: a division into halves of a leaf A_0 gives two sheets A_1, whose division in two gives twice two leaves A_2, etc...

Starting from a paper of format A_4, whose measurements are 21cm x 29.7cm in the normal direction of writing

1) find the measurements of the formats A_n , $n = 0, 1, .., 5.$

2) give the size of the format A_n in general.

Correction

Note by l_n the width and by L_n the length of the format A_n.

1) To get the format A_3 two sheets should be juxtaposed A_4 $(l_4 = 21cm\ ,\ L_4 = 29.7cm)$ (returned widthwise), we get $(l_3 = 29.7cm\ ,\ L_3 = 42cm)$.

Similarly we get
$(l_2 = 42cm\ ,\ L_2 = 59.4cm)$, $(l_1 = 59.4cm\ ,\ L_1 = 84.1cm)$ and $(l_0 = 84.1cm\ ,\ L_0 = 118.9cm)$.

2) Thus by passing from the format A_n in the format A_{n-1} we will have

$$\begin{cases} l_{n-1} = L_n \\ L_{n-1} = 2l_n \end{cases} \quad \text{or} \quad \begin{cases} L_{n+1} = l_n \\ 2\ l_{n+1} = L_n \end{cases}.$$

With $(l_0 = 84.1cm\ ,\ L_0 = 118.9cm)$. Note that the format A_0 is of area $1\ m^2$.

Definitions (Vocabulary):

A sequence $(u_n)_{n\in\mathbb{N}}$ is said to be (from a certain rank)

- **increasing** *if it exists $N \in \mathbb{N}$ such as*
$$n \geq N \implies u_n \leq u_{n+1}.$$
- **Decreasing** *if it exists $N \in \mathbb{N}$ such as*
$$n \geq N \implies u_n \geq u_{n+1}.$$
- **Monotone** *whether it is either increasing or decreasing.*
- **Upper bounded** *if it exists $M \in \mathbb{R}$ such as*
$$n \geq N \implies u_n \leq M;$$
we say that M is an upper bound of the sequence.
- **Lower bounded** *if it exists $m \in \mathbb{R}$ such as*
$$n \geq N \implies u_n \geq m;$$
we says that m is a lower bound of the sequence.
- **bounded** *if it is both upper and lower bounded, i.e. if there is $C \in \mathbb{R}_+$ such as*
$$n \geq N \implies |u_n| \leq C.$$

Notice :

1) A sequence $(u_n)_{n\in\mathbb{N}}$ is increasing if and only if :
$$n \geq N \implies u_{n+1} - u_n \geq 0.$$

2) If $n \geq N \implies u_n > 0$, then $(u_n)_{n\in\mathbb{N}}$ is increasing if and only if :

$$n \geq N \implies \frac{u_{n+1}}{u_n} \geq 1.$$

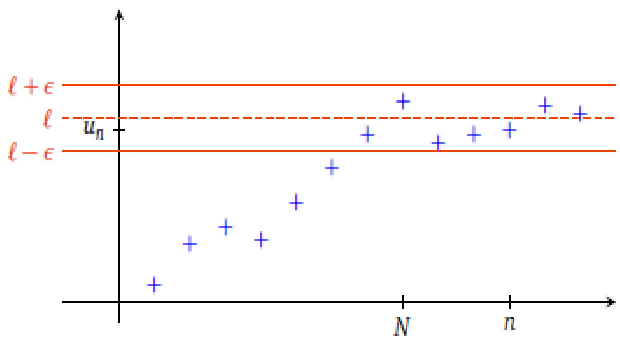
2 CONVERGENCE OF SEQUENCES

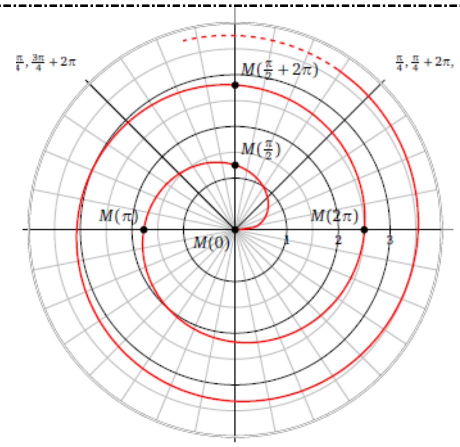
2.1. Limits

Intuitive notion of the limit:

Terms u_n of a sequence of real numbers tend to the number (limit) $l \in \mathbb{R}$ if:

- ***The distance** between the terms of the sequence $(u_n)_{n\in\mathbb{N}}$ and the limit $l \in \mathbb{R}$ is as close as you want from a certain rank.*
- ***Or :** for any neighbourhood of the limit $l \in \mathbb{R}$ (**i.e.** an interval of center l and radius $\varepsilon > 0$) we have all the terms from a certain rank of the sequence $(u_n)_{n\in\mathbb{N}}$ in that neighbourhood.*





✓ **On the left sketch** we present a sequence of real numbers u_n which tends towards $l \in \mathbb{R}$ ($u_n \to l$) when n tends to infinity ($n \to +\infty$).
We notice that for all given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ from which we have the distance between any term $u_n (n > N)$ of the sequence and the limit l ($|u_n - l|$) is less then ε : $$-\varepsilon < u_n - l < +\varepsilon.$$
We also notice that for all given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ from which (i.e. for $n > N$) we have all the terms u_n of the sequence in the **neighbourhood** V_ε **of** l (that is an ***interval cantered at** l **with radius*** ε): $$l - \varepsilon < u_n < l + \varepsilon.$$

✓ **On the right sketch** we present: $M(t)$ which tends to the origin ($M(t) \to O$) when t tends to infinity ($t \to +\infty$).
We notice that for all given $\varepsilon > 0$, there exists $T > 0$ from which we have the distance between any point $M(t)$ ($t > T$) of the sequence and the limit O ($dist\,(M(t) - O)$) is less then ε : $$dist\,(M(t) - O) < \varepsilon.$$

We also notice that for any neighbourhood V_ε of the origin (***disk cantered at the origin and of radius*** $\varepsilon > 0$) it exists $T > 0$ from which from which (i.e. for$t > T$) all points $M(t)$are in the neighbourhood $V_\varepsilon$of the origin : $M(t) \in D(O, \varepsilon)$.

Definition (Finite limit):

The sequence $(u_n)_{n\in\mathbb{N}}$ has for **limit** $l \in \mathbb{R}$ *if:*

for every $\varepsilon > 0$, there is a natural number $N \in \mathbb{N}$ such as

$$n \geq N \implies |u_n - l| < \varepsilon.$$

In other words: *the terms of the sequence $(u_n)_{n\in\mathbb{N}}$ are as close as one wants to$l$ from a certain rank.*

We also say that the sequence $(u_n)_{n\in\mathbb{N}}$ **tends to** $l \in \mathbb{R}$. *We note*

$$\lim_{n\to+\infty} u_n = l \qquad \text{or} \qquad u_n \xrightarrow[n\to+\infty]{} l.$$

Definition (Infinite limit):

The sequence $(u_n)_{n\in\mathbb{N}}$ **tends to** $+\infty$ *if: for every $A > 0$, there is a natural number $N \in \mathbb{N}$ such as* $n \geq N \implies u_n > A$.

The sequence $(u_n)_{n\in\mathbb{N}}$ **tends to** $-\infty$ *if: for every $A > 0$, there is a natural number $N \in \mathbb{N}$ such as* $n \geq N \implies u_n < -A$.

We denote

$$\lim_{n\to+\infty} u_n = +\infty \; (resp. -\infty) \qquad \text{or} \qquad u_n \xrightarrow[n\to+\infty]{} +\infty \; (resp. -\infty).$$

Definition (Convergence, Divergence):

A sequence $(u_n)_{n\in\mathbb{N}}$ is **convergent** *if it admits a* ***finished*** *limit.*

It is **divergent** *if not* [*i.e., either the sequence tends to$\pm\infty$, or it does not admit a limit like $\left((-1)^n\right)_{n\in\mathbb{N}}$*].

Notice: Deleting or modifying a finite number of terms does not modify the **nature** of the sequence (**convergence or divergence**).

We can talk about ***the*** limit if it exists, because there is uniqueness:

Proposition (Unicity of the limit):

If a function admits a limit, then this limit is unique.

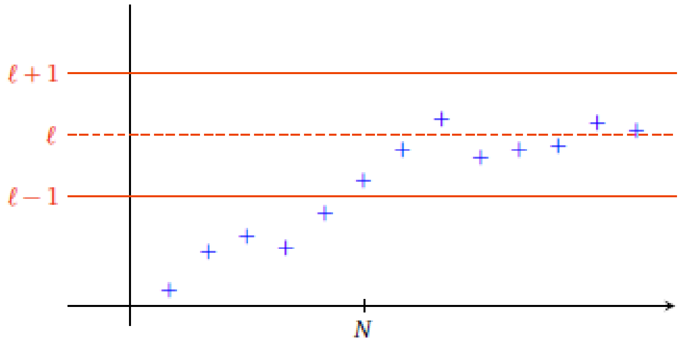
Property (Operations and Limits):
Let be $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ two sequences such that
$$x_n \xrightarrow[n\to+\infty]{} x \quad \text{and} \quad y_n \xrightarrow[n\to+\infty]{} y \quad \text{with } -\infty \le x, y \le +\infty,$$
then agreeing in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$ we have :

- $x_n + y_n \xrightarrow[n\to+\infty]{} x + y$ *except for the case* $-\infty + \infty$ *(which is an indeterminate case).*
- $x_n \times y_n \xrightarrow[n\to+\infty]{} x \times y$ *except for the case* $0 \times \infty$ *(which is an indeterminate case).*
- $\forall \lambda \in \mathbb{R} : (\lambda x_n) \xrightarrow[n\to+\infty]{} \lambda x$.
- *If* $x \neq 0$ *and* $x_n \neq 0$ *from a certain rank then* $\dfrac{1}{x_n} \xrightarrow[n\to+\infty]{} \dfrac{1}{x}$.
- *If* $y \neq 0$ *and* $y_n \neq 0$ *from a certain rank then*
  $\dfrac{x_n}{y_n} \xrightarrow[n\to+\infty]{} \dfrac{x}{y}$ *except for cases* $\dfrac{0}{0}$ *or* $\dfrac{\infty}{\infty}$ *(undetermined cases).*

Theorem (Necessary condition)
Any convergent sequence is bounded.

An unbounded sequence cannot be convergent.

To see that, it will be enough to take $\varepsilon = 1$.



Limits to remember:

- $$(a \in \mathbb{R}) \ , \ \lim_{n\to+\infty} a^n = \begin{cases} +\infty & \text{si } a > 1 \\ 1 & \text{si } a = 1 \\ 0 & \text{si } -1 < a < +1 \\ \textit{pas de limite} & \text{si } a \le -1 \ . \end{cases}$$
- $$\lim_{n\to+\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

- $(a \in \mathbb{R})$, $\lim\limits_{n\to+\infty} \left(1+\dfrac{a}{n}\right)^n = \lim\limits_{n\to+\infty} e^{n \ln\left(1+\frac{a}{n}\right)} = \lim\limits_{n\to+\infty} e^{a \frac{\ln\left(1+\frac{a}{n}\right)}{\frac{a}{n}}} = e^a.$
- *For*$(\alpha \in \mathbb{R})$

$$\lim_{n\to+\infty} \sqrt[n]{n} = 1 \quad , \quad \lim_{n\to+\infty} n^\alpha = \begin{cases} 0 & \text{si } \alpha < 0 \\ 1 & \text{si } \alpha = 0 \\ +\infty & \text{si } \alpha > 0\,. \end{cases}$$

Examples

1) $\lim\limits_{n\to+\infty} \left(1-\dfrac{1}{n}\right)^n = e^{-1} = \dfrac{1}{e}$, **2)** $\lim\limits_{n\to+\infty} \left(1+\dfrac{1}{n}\right)^{n^2} = \lim\limits_{n\to+\infty} \left[\left(1+\dfrac{1}{n}\right)^n\right]^n = +\infty$

3) $\lim\limits_{n\to+\infty} \left(1+\dfrac{1}{n^2}\right)^n = \lim\limits_{n\to+\infty} \left[\left(1+\dfrac{1}{n^2}\right)^{n^2}\right]^{\frac{1}{n}} = e^0 = 1.$

Exercise

Consider a regular n-sided polygon inscribed in a disk of radius r. Show that its perimeter tends to the length of the circle as n tends to infinity.

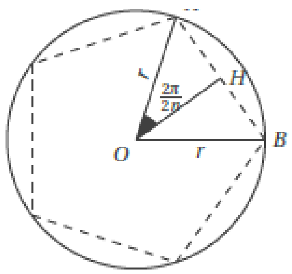
Correction

$AB = 2\,AH = 2\,r \sin\left(\frac{2\pi}{2n}\right)$ so, the perimeter is

$$p(n) = n\,AB = 2n\,r \sin\left(\frac{\pi}{n}\right).$$

Then we have

$$\lim_{n\to+\infty} p(n) = \lim_{n\to+\infty} 2\pi r \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} = \lim_{n\to+\infty} \frac{\sin\dfrac{1}{n}}{\dfrac{1}{n}} = 2\pi r.$$



Exercise

Calculate, if they exist, the following limits:

1) $\lim\limits_{n\to+\infty} n \sin\dfrac{1}{n}$, **2)** $\lim\limits_{n\to+\infty} \dfrac{\sin\dfrac{1}{n}}{n}$, **3)** $\lim\limits_{n\to+\infty} \dfrac{\sin n}{n}$, **4)** $\lim\limits_{n\to+\infty} n \sin(n)$.

5)$u_n = \left(1+\dfrac{2}{n}\right)^n$, **6)**$u_n = \left(1+\dfrac{2}{\sqrt{n}}\right)^n$, **7)**$u_n = \left(1+\dfrac{2}{n}\right)^{\sqrt{n}}$, **8)**$u_n = \left(1+\dfrac{(-1)^n}{n}\right)^n$,

9)$u_n = \left(1+\dfrac{3}{n}\right)^{4n}$, **10)**$u_n = \dfrac{\left(3+\frac{1}{n}\right)^{2n}}{9^n}$.

Correction

1) $\lim\limits_{n\to+\infty} n \sin\dfrac{1}{n} = \lim\limits_{n\to+\infty} \dfrac{\sin\dfrac{1}{n}}{\dfrac{1}{n}} = \lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1.$

2) $\lim\limits_{n\to+\infty} \dfrac{\sin\dfrac{1}{n}}{n} = 0.$

3) $\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{+1}{n}$ **and** $\lim\limits_{n\to+\infty} \frac{-1}{n} = \lim\limits_{n\to+\infty} \frac{+1}{n} = 0$ **, using the frame theorem we deduce** $\lim\limits_{n\to+\infty} \frac{\sin(n)}{n} = 0.$

4) $\lim\limits_{n\to+\infty} n\,\sin(n)$**:** **If** $n = \frac{\pi}{2} + 2k\pi$ **we have** $\lim\limits_{n\to+\infty} n\,\sin(n) = +\infty$ **and if** $n = \frac{3\pi}{2} + 2k\pi$ **we have** $\lim\limits_{n\to+\infty} n\,\sin(n) = -\infty$**. Therefore, we have two distinct adherent points, so** $\lim\limits_{n\to+\infty} n\,\sin(n)$ **does not exist.**

5) $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} (1+\frac{2}{n})^n = e^2.$

6) $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} (1+\frac{2}{\sqrt{n}})^{\sqrt{n}\times\sqrt{n}} = \lim\limits_{n\to+\infty} \left[(1+\frac{2}{\sqrt{n}})^{\sqrt{n}}\right]^{\sqrt{n}} = +\infty.$

7) $u_n = (1+\frac{2}{n})^{\sqrt{n}}$**:** $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} (1+\frac{2}{n})^{\frac{n}{\sqrt{n}}} = \lim\limits_{n\to+\infty} \left[(1+\frac{2}{n})^n\right]^{\frac{1}{\sqrt{n}}} = 0.$

8) $u_n = \left(1+\frac{(-1)^n}{n}\right)^n$ **: whether** $n = 2k$ **we have** $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} (1+\frac{1}{n})^n = e$ **and if** $n = 2k+1$ **we have** $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} (1+\frac{-1}{n})^n = e^{-1}$**; so** $\lim\limits_{n\to+\infty} u_n$ **does not exist.**

9) $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} (1+\frac{3}{n})^{4n} = \lim\limits_{n\to+\infty} \left[(1+\frac{3}{n})^n\right]^4 = (e^3)^4 = e^{12}.$

10) $u_n = \frac{(3+\frac{1}{n})^{2n}}{9^n} = \frac{3^{2n}\,(1+\frac{1}{3n})^{2n}}{3^{2n}} = \left[(1+\frac{1/3}{n})^n\right]^2 = e^{\frac{2}{3}}$

2.2. Comparison

Theorem (Comparisons):

Let be two sequences $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *convergent satisfying (from a certain rank)* $x_n \leq y_n$ *then*

$$\lim_{n\to+\infty} x_n \leq \lim_{n\to+\infty} y_n.$$

In particular I f $\lim\limits_{n\to+\infty} x_n = +\infty$ *then* $\lim\limits_{n\to+\infty} y_n = +\infty.$

Attention: after passing to the limit, strict inequalities become wide inequalities:

$$\forall n \in \mathbb{N} : u_n > 0 \implies \lim_{n\to+\infty} u_n \geq 0.$$

Example: $\forall n \in \mathbb{N} : u_n = \frac{1}{n} > 0$ but $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} \frac{1}{n} = 0.$

Theorem (frame or gendarmes):

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence. If there are two convergent sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ such that:

1) $x_n \leq u_n \leq y_n$ *(from a certain rank)* ***2)*** $\lim\limits_{x\to+\infty} x_n = \lim\limits_{x\to+\infty} y_n = l \in \mathbb{R}$

then $(u_n)_{n\in\mathbb{N}}$ converges and we have $\lim\limits_{n\to+\infty} u_n = l$.

Example

We have $\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{+1}{n}$ with $\lim\limits_{n\to+\infty} \frac{-1}{n} = \lim\limits_{n\to+\infty} \frac{+1}{n} = 0$,

we deduce applying the theorem that $\lim\limits_{n\to+\infty} \frac{\sin(n)}{n} = 0$.

Corollary:

If $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence and $(y_n)_{n\in\mathbb{N}}$ converges to $y = 0$ then the sequence $(x_n \times y_n)_{n\in\mathbb{N}}$ converges and we have $\lim\limits_{n\to+\infty} (x_n \times y_n) = 0$.

Example

If $(x_n)_{n\in\mathbb{N}}$ is the sequence given by $x_n = \cos(n)$ and $(y_n)_{n\in\mathbb{N}}$ is defined by $y_n = \frac{1}{n}$, then $\lim\limits_{n\to+\infty} \frac{\cos(n)}{n} = 0$ because $0 \leq \left|\frac{\cos(n)}{n}\right| \leq \frac{1}{n} \to 0$.

Exercise

Answer true or false, justifying your answer.

1. If a sequence $(|u_n|)$is upper-bounded, then (u_n) is bounded.
2. If a sequence $(|u_n|)$ converges to 0, then the sequence (u_n) converges to 0.
3. If a sequence (u_n) converges to l and if it is in strictly positive terms, then $l > 0$.
4. If a sequence (u_n) converges to 0, then the sequence $(u_n \times v_n)$ converges to 0 whatever is the sequence (v_n).
5. If$(|u_n|)$ and $(|v_n|)$ are two convergent sequences, the sequence $(|u_n + v_n|)$ is also convergent.
6. If the result $(|u_n + v_n|)$ is convergent then the two sequences $(|u_n|)$ and $(|v_n|)$ are also convergent.
7. If (u_n) and (v_n) are two sequences such that from a certain rank we have $u_n \leq v_n$, then the convergence of (v_n) implies convergence of (u_n).
8. If a sequence (u_n) is monotone, then it converges.
9. If a sequence (u_n) is monotonic and upper-bounded, then it converges.

Correction

1. True: if there exists a positive real a such that $|u_n| \leq a$ has then $-a \leq u_n \leq +a$ therefore the sequence (u_n) is bounded.
2. True: $-|u_n| \leq u_n \leq |u_n|$, we obtain the result by applying the gendarme's theorem.
3. False: $u_n = 1/n > 0$ for all n but $u_n \to l = 0$.
4. False: counterexample $u_n = 1/n \to 0$ and $v_n = n$ *but* $u_n \times v_n = 1 \not\to 0$.
5. True Because if $|u_n| \to l_1$ and $|v_n| \to l_2$, we have $|u_n + v_n| \leq |u_n| + |v_n|$ so applying the comparison theorem we deduce that $(|u_n + v_n|)$ is convergent to a limit $l \leq l_1 + l_2$.
6. False: counterexample, for $u_n = n$ and $v_n = -n$ *we have* $|u_n + v_n| = 0 \to 0$ while both $(|u_n|)$ and $(|v_n|)$ diverge.
7. False: for $u_n = -n$ and $v_n = 1/n$ we have $u_n \leq 0 \leq v_n$, but (v_n) converges while (u_n) diverges.
8. False: $u_n = -n$ *is monotonic but divergent.*
9. Wrong: If the sequence (u_n) is monotone then two cases arise: (u_n) is increasing then it converges since it is also upper-bounded; (u_n) is decreasing then nothing can be said about its convergence.

Exercise

Calculate, if they exist, the limits of the sequences $(u_n)_{n\in\mathbb{N}}$following: $\forall n \in \mathbb{N}^*$

1) $u_n > \ln(n)$, **2)** $\dfrac{1}{n+1} \leq u_n \leq \dfrac{1}{n}$, **3)** $u_0 < 1, u_n \nearrow, u_n < 1 + \dfrac{1}{n}$, **4)** $u_n = \ln(n) + \sin(n)$,

5) $u_n = \sin(\dfrac{n\pi}{3})$, **6)** $u_n = \dfrac{n\sin(n)}{n^2+1}$, **7)** $u_n = \sin(\pi + \dfrac{\pi}{n})$.

Correction

1) $\lim\limits_{n\to+\infty} u_n \geq \lim\limits_{n\to+\infty} \ln(n) = +\infty$ so $\lim\limits_{n\to+\infty} u_n = +\infty$.

2).$\dfrac{1}{n+1} \leq u_n \leq \dfrac{1}{n}$ and $\lim\limits_{n\to+\infty} \dfrac{1}{n+1} = \lim\limits_{n\to+\infty} \dfrac{1}{n} = 0$, from the frames theorem we deduce $\lim\limits_{n\to+\infty} u_n = 0$.

3) $u_n \nearrow$ and $\forall n \in \mathbb{N}^* : u_n < 1 + \dfrac{1}{n} < 2$ being an increasing and upper-bounded sequence, it converges.

Moreover, we have $\lim\limits_{n\to+\infty} u_n \leq \lim\limits_{n\to+\infty} 1 + \dfrac{1}{n} = 1$.

4).$\ln(n) - 1 \leq u_n = \ln(n) + \sin(n) \leq \ln(n) + 1$ and $\lim\limits_{n\to+\infty} \ln(n) - 1 = \lim\limits_{n\to+\infty} \ln(n) + 1 = +\infty$, we deduce $\lim\limits_{n\to+\infty} u_n = +\infty$.

5) $\lim\limits_{n\to+\infty} \sin(\dfrac{n\pi}{3})$: If $n = 3k$ we have $\lim\limits_{n\to+\infty} \sin(\dfrac{n\pi}{3}) = \lim\limits_{n\to+\infty} \sin(k\pi) = 0 \to 0$ and if $n = 6k\pi + 1$we obtain $\lim\limits_{n\to+\infty} \sin(\dfrac{n\pi}{3}) = \lim\limits_{n\to+\infty} \sin(2k\pi + \dfrac{\pi}{3}) = \sqrt{3}/2 \not\to 0$. We have two subsequences with two distinct limits, therefore $\lim\limits_{n\to+\infty} \sin(\dfrac{n\pi}{3})$ **does not exist.**

6). $\frac{-n}{n^2+1} \leq u_n = \frac{n\sin(n)}{n^2+1} \leq \frac{+n}{n^2+1}$ **and** $\lim_{n\to+\infty}\frac{\pm n}{n^2+1} = \lim_{n\to+\infty}\frac{\pm 1}{n} = 0$, **we deduce** $\lim_{n\to+\infty} u_n = 0$.

7) $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \sin(\pi + \frac{\pi}{n}) = \sin(\pi) = 0.$

Definition (equivalent sequences, negligible sequence with respect to ...)

- *Two sequences* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ (*non-zero from a certain rank*) *are said to be* **equivalent** (*near infinity*) *if* $\lim_{n\to+\infty}\frac{a_n}{b_n} = 1$. *We note* $a_n \sim b_n$.
- $(a_n)_{n\in\mathbb{N}}$ *is said to be* **negligible compared to** $(b_n)_{n\in\mathbb{N}}$ *if* $\lim_{n\to+\infty}\frac{a_n}{b_n} = 0$. *We denote* $a_n = \circ(b_n)$.

2.2.1. Comparisons to remember

1) If $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ a polynomial function, then

$$P(x) \underset{+\infty}{\sim} a_nx^n.$$

2) If $F(x) = \frac{a_0 + a_1x + a_2x^2 + \ldots + a_nx^n}{b_0 + b_1x + b_2x^2 + \ldots + b_px^p}$ $(a_n, b_p \neq 0)$ a rational function, then

$$F(x) \underset{+\infty}{\sim} \frac{a_nx^n}{b_px^p}.$$

3) Trigonometric functions:

$$\sin x \underset{0}{\sim} x\,, \qquad \tan x \underset{0}{\sim} x \qquad , \qquad 1-\cos x \underset{0}{\sim} \frac{x^2}{2}.$$

4) Logarithm, exponential, power functions

$$\ln(1+x) \underset{0}{\sim} x, e^x - 1 \underset{0}{\sim} x, (1+x)^\alpha - 1 \underset{0}{\sim} \alpha\, x.$$

5) For$(k>0)$ (**"exponential base cste << factorial << exponential base var."**)

$$\lim_{n\to+\infty}\frac{k^n}{n!} = 0 \;\; i.e. \;\; k^n \underset{+\infty}{=} o(n!) \quad , \quad \lim_{n\to+\infty}\frac{n!}{n^n} = 0 \;\; i.e. \;\; n! \underset{+\infty}{=} o(n^n).$$

6) For$(k>0)$, $\alpha > 0$ (**"logarithm << power << exponential"**)

$$\lim_{n\to+\infty}\frac{(\ln n)^\beta}{n^\alpha} = 0 \;\; i.e. \;\; (\ln n)^\beta \underset{+\infty}{=} o(n^\alpha) \quad , \quad \lim_{n\to+\infty}\frac{n^\alpha}{k^n} = 0 \;\; i.e. \;\; n^\alpha \underset{+\infty}{=} o(k^n).$$

Examples

1) $\lim\limits_{n\to+\infty} n^2\left(1-\cos\dfrac{1}{n}\right) = \lim\limits_{n\to+\infty} \dfrac{1-\cos\dfrac{1}{n}}{\dfrac{1}{n^2}} = \lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}.$

2) $\lim\limits_{n\to+\infty} n\,\ln\left(1+\dfrac{1}{n}\right) = \lim\limits_{n\to+\infty} \dfrac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \lim\limits_{x\to 0} \dfrac{\ln\left(1+x\right)}{x} = 1.$

Exercise

Calculate, if they exist, the limits of the sequences $(u_n)_{n\in\mathbb{N}}$:$\forall n\in\mathbb{N}^*$

1) $u_n = \sqrt[n]{n}$, **2)** $u_n = \dfrac{n}{e}+\dfrac{1}{e^n}$, **3)** $u_n = \dfrac{n}{n+1}\ln(n)$, **4)** $u_n = \dfrac{n^2}{n!}$,

5)$u_n = \dfrac{(2n)!}{n!}$, **6)** $u_n = \dfrac{n^2}{n^2+1}$, **7)** $u_n = (-1)^n\dfrac{n^2}{n^2+1}$, **8)** $u_n = \dfrac{n+(-1)^n}{3n-(-1)^n}$,

9) $u_n = \dfrac{n^2+1}{2n\sqrt{n}+1}$ **10)** $u_n = \sqrt{3n+1}-\sqrt{2n+1}$.

Correction

1)$u_n = \sqrt[n]{n} = n^{\frac{1}{n}} = e^{\frac{\ln(n)}{n}} \to e^0 = 1$ **because** $\ln(n) \underset{+\infty}{=} o(n)$.

2) $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty}\dfrac{n}{e} + \lim\limits_{n\to+\infty}\dfrac{1}{e^n} = +\infty.$

3) $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty}\dfrac{n}{n+1}\ln(n) = +\infty$ **because** $\dfrac{n}{n+1}\to 1.$

4) we have $n^\alpha \underset{+\infty}{=} o(k^n)$ **and** $k^n \underset{+\infty}{=} o(n!)$ **so** $n^\alpha \underset{+\infty}{=} o(n!)$, **we deduce**

$\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty}\dfrac{n^2}{n!} = 0$

5)$u_n = \dfrac{(2n)!}{n!} = \dfrac{(2n)(2n-1)..(n+1)n!}{n!} = (2n)(2n-1)..(n+1) \to +\infty$

6)$u_n = \dfrac{n^2}{n^2+1} \sim \dfrac{n^2}{n^2} = 1$ **then** $\lim\limits_{n\to+\infty} u_n = 1.$

7)$u_n = (-1)^n\dfrac{n^2}{n^2+1}$**: if** $n=2k$ **we have** $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty}\dfrac{(2k)^2}{(2k)^2+1} = +1$**and if** $n=2k+1$

we have $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} -\dfrac{(2k+1)^2}{(2k+1)^2+1} = -1$**. Therefore** $\lim\limits_{n\to+\infty} u_n$ **does not exist.**

8) $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty}\dfrac{n+(-1)^n}{3n-(-1)^n} = \lim\limits_{n\to+\infty}\dfrac{n}{3n} = \dfrac{n}{3n}.$

9)$u_n = \dfrac{n^2+1}{2n\sqrt{n}+1} \sim \dfrac{n^2}{2n\sqrt{n}} = \dfrac{\sqrt{n}}{2}$, **we deduce** $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty}\dfrac{\sqrt{n}}{2} = +\infty$

10)

$$u_n = \sqrt{3n+1}-\sqrt{2n+1} = \frac{\left(\sqrt{3n+1}-\sqrt{2n+1}\right)\left(\sqrt{3n+1}+\sqrt{2n+1}\right)}{\sqrt{3n+1}+\sqrt{2n+1}} = \frac{(3n+1)-(2n+1)}{\sqrt{3n+1}+\sqrt{2n+1}}$$

$$= \frac{n}{\sqrt{3n+1}+\sqrt{2n+1}} \sim \frac{n}{\sqrt{3n}+\sqrt{2n}} = \frac{\sqrt{n}}{\sqrt{3}+\sqrt{2}} \to +\infty\ .$$

Exercise

Calculate, if they exist, the limits of the sequences $(u_n)_{n\in\mathbb{N}}$: $\forall n \in \mathbb{N}^*$

1)$u_n = n^2\big[1-\cos\big(\frac{2}{n}\big)\big]$, **2)**$u_n = n^2\big[1-\cos\big(\frac{2}{\sqrt{n}}\big)\big]$, **3)**$u_n = n\big[1-\cos\big(\frac{2}{n}\big)\big]$,

4)$u_n = n^3\big[\tan\big(\frac{3}{n}\big) - \sin\big(\frac{3}{n}\big)\big]$.

Correction

1)we have$1-\cos x \underset{0}{\sim} \frac{x^2}{2}$**from where**

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} n^2\ \Big[1-\cos\Big(\frac{2}{n}\Big)\Big] = \lim_{n\to+\infty} n^2\ \frac{\big(\frac{2}{n}\big)^2}{2} = \lim_{n\to+\infty} \frac{4\,n^2}{2\,n^2} = 2.$$

2)$u_n = n^2\big[1-\cos\big(\frac{2}{\sqrt{n}}\big)\big] \sim n^2\ \frac{\big(\frac{2}{\sqrt{n}}\big)^2}{2} = \frac{4\,n^2}{2\,n} = 2n$, **from where**

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} n^2\ \Big[1-\cos\Big(\frac{2}{\sqrt{n}}\Big)\Big] = \lim_{n\to+\infty} 2n = +\infty$$

3)$u_n = n\big[1-\cos\big(\frac{2}{n}\big)\big] \sim n\ \frac{\big(\frac{2}{n}\big)^2}{2} = \frac{4\,n}{2\,n^2} = \frac{2}{n}$, **from where** $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} \frac{2}{n} = 0$.

4)$u_n = n^2\big[(-1)^n - \cos\big(\frac{1}{n}\big)\big]$**: whether**$\left\{ \begin{smallmatrix} l_{n-1} = L_n \\ L_{n-1} = 2l_n \end{smallmatrix}\right.$**we have**

$$u_n = (2k)^2\Big[1-\cos\Big(\frac{1}{2k}\Big)\Big] \sim (2k)^2\ \frac{\big(\frac{1}{2k}\big)^2}{2} = \frac{1}{2} \to \frac{1}{2}$$

And if$n = 2k+1$ **we have**$u_n = (2k+1)^2\big[-1-\cos\big(\frac{1}{2k}\big)\big] \to -\infty$. $\lim\limits_{n\to+\infty} u_n$**does not exist.**

4)$u_n = n^3\big[\tan\big(\frac{3}{n}\big) - \sin\big(\frac{3}{n}\big)\big] = n^3\ \tan\big(\frac{3}{n}\big)\big[1-\cos\big(\frac{3}{n}\big)\big]$; **we have**$1-\cos x \underset{0}{\sim} \frac{x^2}{2}$**And**

$\tan x \underset{0}{\sim} x$**we deduce**$u_n \sim n^3\ \frac{3}{n}\frac{\big(\frac{3}{n}\big)^2}{2} = \frac{27}{2}$**from where** $\lim\limits_{n\to+\infty} u_n = \frac{27}{2}$.

Exercise

Calculate, if they exist, the limits of the sequences$(u_n)_{n\in\mathbb{N}}$following:$\forall n \in \mathbb{N}^*$

1)$u_n = \sqrt{n^5+3n} - n$, **2)**$u_n = n - \sqrt{n^3-3n}$, **3)**$u_n = n - \sqrt{n^2-3n}$,

4)$u_n = \sum\limits_{k=1}^{k=n} \frac{1}{\sqrt{n^2+k}}$.

Correction

1)$u_n = \sqrt{n^5+3n} - n$

$u_n = \frac{\big(\sqrt{n^5+3n}-n\big)\big(\sqrt{n^5+3n}+n\big)}{\sqrt{n^5+3n}+n} = \frac{n^5+3n-n^2}{n^{\frac{5}{2}}\sqrt{1+\frac{3}{n^4}}+n} \sim \frac{n^5}{n^{\frac{5}{2}}} = n^{\frac{5}{2}}$, **we deduce** $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} n^{\frac{5}{2}} = +\infty$.

2) $u_n = n - \sqrt{n^3 - 3n}$

$u_n = \dfrac{\left(n - \sqrt{n^3 - 3n}\right)\left(n + \sqrt{n^3 - 3n}\right)}{n + \sqrt{n^3 - 3n}} = \dfrac{n^2 - (n^3 - 3n)}{n + n^{\frac{3}{2}}\sqrt{1 - \frac{3}{n^2}}} \sim \dfrac{-n^3}{n^{\frac{3}{2}}} = -n^{\frac{3}{2}}$, we deduce $\lim\limits_{n \to +\infty} u_n = \lim\limits_{n \to +\infty} -n^{\frac{3}{2}} = -\infty$.

3) $u_n = n - \sqrt{n^2 - 3n}$

$u_n = \dfrac{\left(n - \sqrt{n^2 - 3n}\right)\left(n + \sqrt{n^2 - 3n}\right)}{n + \sqrt{n^2 - 3n}} = \dfrac{n^2 - (n^2 - 3n)}{n + n^{\frac{3}{2}}\sqrt{1 - \frac{3}{n^2}}} \sim \dfrac{3n}{n^{\frac{3}{2}}} = 3n^{-\frac{1}{2}}$, we deduce $\lim\limits_{n \to +\infty} u_n = \lim\limits_{n \to +\infty} 3n^{-\frac{1}{2}} = 0$.

4) $u_n = \sum\limits_{k=1}^{k=n} \dfrac{1}{\sqrt{n^2 + k}}$. For every $1 \le k \le n$ we have $\dfrac{1}{\sqrt{n^2 + n}} \le \dfrac{1}{\sqrt{n^2 + k}} \le \dfrac{1}{\sqrt{n^2 + 1}}$,

adding term by term we get $\dfrac{n}{\sqrt{n^2 + n}} \le u_n = \sum\limits_{k=1}^{k=n} \dfrac{1}{\sqrt{n^2 + k}} \le \dfrac{n}{\sqrt{n^2 + 1}}$.

On the other hand $\dfrac{n}{\sqrt{n^2 + n}} \sim \dfrac{n}{\sqrt{n^2}} = 1$ and $\dfrac{n}{\sqrt{n^2 + 1}} \sim \dfrac{n}{\sqrt{n^2}} = 1$ so

$\lim\limits_{n \to +\infty} \dfrac{n}{\sqrt{n^2 + n}} = \lim\limits_{n \to +\infty} \dfrac{n}{\sqrt{n^2 + 1}} = 1$,

we deduce from the gendarmes theorem that $\lim\limits_{n \to +\infty} u_n = 1$.

2.3. Extracted sequence

2.3.1. Definition and properties

Definition (Extracted sequence or sub-sequence, adhesion value)

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence. We call extracted sequence or sub-sequence of $(u_n)_{n\in\mathbb{N}}$ every sequence $(v_n)_{n\in\mathbb{N}}$ defined by

$$v_n = u_{\varphi(n)}, \qquad \text{where } \varphi : \mathbb{N} \to \mathbb{N} \text{ is a strictly increasing map.}$$

We say that $l \in \mathbb{R}$ is an **adherent point** *of the sequence $(u_n)_{n\in\mathbb{N}}$ if there is an extracted subsequence that converges to l.*

Example

- ✓ *$(u_{2n})_{n\in\mathbb{N}}$ and $(u_{2n+1})_{n\in\mathbb{N}}$ are subsequences of the sequence $(u_n)_{n\in\mathbb{N}}$.*
- ✓ *$(u_{3n})_{n\in\mathbb{N}}$, $(u_{3n+1})_{n\in\mathbb{N}}$ and $(u_{3n+2})_{n\in\mathbb{N}}$ are sequences extracted from the sequence$(u_n)_{n\in\mathbb{N}}$.*
- ✓ *Consider the sequence $\left(u_n = (-1)^n\right)_{n\in\mathbb{N}}$: $(u_{2n} = 1)_{n\in\mathbb{N}}$ is an extracted sequence with limit 1 and $(u_{2n+1} = -1)_{n\in\mathbb{N}}$ is an extracted sequence with limit -1.*
  Therefore, $\left(u_n = (-1)^n\right)_{n\in\mathbb{N}}$ has two adherent points, -1 and $+1$.

Exercise

Determine the adherent points of the sequence $(u_n)_{n\in\mathbb{N}^*}$ in the following cases.

$$\textbf{1) } u_n = \frac{(-1)^n}{1+\frac{1}{n}} \qquad\qquad \textbf{2) } u_n = \sin\left(\frac{n\,\pi}{2}\right).$$

Correction

1) $u_{2n} = \dfrac{+1}{1+\frac{1}{2n}} \to 1$ and $u_{2n+1} = \dfrac{-1}{1+\frac{1}{2n}} \to -1$, hence the set of adherent points is $\{-1, 1\}$.

2) $u_0 = 0, u_1 = 1, u_2 = 0, u_3 = -1$ and the values repeat ... so we can deduce that $u_{4n} = 0 \to 0$, $u_{4n+1} = 1 \to 1$ and $u_{4n+3} = -1 \to -1$, hence the set of adherent points is $\{-1, 0, 1\}$.

Theorem

Let be $(u_n)_{n\in\mathbb{N}}$ a sequence.

- *If $\lim\limits_{n\to+\infty} u_n = l$ then any subsequence converges also to the limit l.*
- *If a sequence extracted from $(u_n)_{n\in\mathbb{N}}$ diverges then $(u_n)_{n\in\mathbb{N}}$ diverges.*
- *If two sequences extracted from $(u_n)_{n\in\mathbb{N}}$ have different limits (two different adherent points) then $(u_n)_{n\in\mathbb{N}}$ diverges.*

- *If we can decompose $(u_n)_{n\in\mathbb{N}}$ into two (or more) extracted sequences converging towards the same limit $l \in \mathbb{R}$ then $(u_n)_{n\in\mathbb{N}}$ converges also to the limit l.*
  For example, if $(u_{2n})_{n\in\mathbb{N}}$ and $(u_{2n+1})_{n\in\mathbb{N}}$ converge to the same limit l, then $(u_n)_{n\in\mathbb{N}}$ converges to l.
  [the same frame if we decompose $(u_n)_{n\in\mathbb{N}}$ in $(u_{3n})_{n\in\mathbb{N}}$, $(u_{3n+1})_{n\in\mathbb{N}}$ and $(u_{3n+2})_{n\in\mathbb{N}}$].

Example

- ✓ *The sequence $\left(u_n = \dfrac{1}{n}\right)_{n\in\mathbb{N}}$ converges to $l = 0$, so 0 is the unique adherence value of the sequence $(u_n)_{n\in\mathbb{N}}$.*
- ✓ *The sequence $\left(u_n = (-1)^n\right)_{n\in\mathbb{N}}$ admits two different adherent points : -1 (limit of $(u_{2n} = 1)_{n\in\mathbb{N}}$) and $+1$ (limit of $(u_{2n} = 1)_{n\in\mathbb{N}}$), so it diverges.*

- ✓ *Consider the sequence* $\left(u_n = \frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$: $\left(u_{2n} = \frac{+1}{n}\right)_{n\in\mathbb{N}}$ *and* $\left(u_{2n+1} = \frac{-1}{n}\right)_{n\in\mathbb{N}}$ *converge to the same limit* $l = 0$, *then* $(u_n)_{n\in\mathbb{N}}$ *converges to* 0 *which is the unique adherent point of* $\left(u_n = \frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$.

2.3.2. Adjacent sequences

Definition (Adjacent sequences)

The sequences $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *are said to be* **adjacent** *if*

1) $(a_n)_{n\in\mathbb{N}}$ *is increasing,* **2)** $(b_n)_{n\in\mathbb{N}}$ *is decreasing* **2)** $\lim\limits_{n\to+\infty}(a_n - b_n) = 0$.

Theorem (of monotone convergence):

- *If a sequence* $(u_n)_{n\in\mathbb{N}}$ *is* ***increasing*** *and* ***upper-bounded*** *then it* ***converges*** *and* $\lim\limits_{n\to+\infty} u_n = \sup\limits_{n\in\mathbb{N}} u_n$.
- *If a sequence* $(u_n)_{n\in\mathbb{N}}$ *is increasing and not bounded then it diverges towards* $+\infty$.
- *If a sequence* $(u_n)_{n\in\mathbb{N}}$ *is* ***decreasing*** *and* ***lower-bounder*** *then it* ***converges*** *and* $\lim\limits_{n\to+\infty} u_n = \inf\limits_{n\in\mathbb{N}} u_n$.
- *If a sequence* $(u_n)_{n\in\mathbb{N}}$ *is decreasing and not lowered then it diverges towards* $-\infty$.

Theorem

If two sequences are adjacent then they converge and have the same limit.

Indeed, the terms of the sequences are ordered as follows:

$$a_0 \le a_1 \le a_2 \le \dots \le a_n \le \dots \dots \le b_n \le \dots \le b_2 \le b_1 \le b_0.$$

The *sequence* $(a_n)_{n\in\mathbb{N}}$ is increasing and upper-bounded by b_0, so it converges to a limit l.

The *sequence* $(b_n)_{n\in\mathbb{N}}$ is decreasing and lower-bounded by a_0, so it converges to a limit l'.

Being adjacent, we have $0 = \lim\limits_{n\to+\infty}(a_n - b_n) = l - l'$ which means $l = l'$.

Example: Let be $a_n = 1 - \frac{1}{n}$ and $b_n = 1 + \frac{1}{n^2}$.

$(a_n)_{n\in\mathbb{N}}$ *is increasing because* $a_{n+1} - a_n = -\frac{1}{n+1} + \frac{1}{n} = \frac{1}{n(n+1)} > 0$,

$(b_n)_{n\in\mathbb{N}}$ *is decreasing because* $b_{n+1} - b_n = \frac{1}{(n+1)^2} - \frac{1}{n^2} = \frac{-2n-1}{n^2(n+1)^2} < 0$,

and $\lim\limits_{n\to+\infty}(a_n - b_n) = \lim\limits_{n\to+\infty}(-\frac{1}{n} - \frac{1}{n^2}) = 0.$

So $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *are adjacent and, applying the theorem,* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ converge and *we have* $\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} b_n$.

Indeed, by a direct calculation, we have $\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} b_n = 1.$

Example: Consider $S_n = \sum\limits_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$. We want to show that $(S_n)_{n\in\mathbb{N}^*}$converges. For this we introduce the two sequences $a_n = S_n$ and $b_n = S_n + \frac{1}{n}$ then we prove that they are adjacent.

$(a_n)_{n\in\mathbb{N}}$ *is increasing because*

$$a_{n+1} - a_n = S_{n+1} - S_n = \textstyle\sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{(k+1)^2} > 0,$$

$(b_n)_{n\in\mathbb{N}}$ *is decreasing because*

$$b_{n+1} - b_n = S_{n+1} + \frac{1}{n+1} - S_n - \frac{1}{n} = +\frac{1}{n+1} = \frac{1}{(k+1)^2} + \frac{1}{n+1} - \frac{1}{n} = -\frac{-1}{n(n+1)^2} < 0,$$

and $\lim\limits_{n\to+\infty}(a_n - b_n) = \lim\limits_{n\to+\infty} \frac{1}{n} = 0.$

So $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *are adjacent and, by the theorem, the two sequences converge. Especially* $(a_n)_{n\in\mathbb{N}} = (S_n)_{n\in\mathbb{N}}$ *converges i.e.*

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \dots < \infty.$$

We say that the serie (infinite sum) $\sum\limits_{k=1}^{+\infty} \frac{1}{k^2}$ converges.

Exercise: ***(*****Arithmetic-harmonic mean)**

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences such that $a_0 > b_0 > 0$ and $\forall n \in \mathbb{N}$

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad \text{and} \qquad$$ [illegible]

1. Check that the sequence $(a_n \times b_n)_{n\in\mathbb{N}}$ is stationary.
2. Assuming it exists, calculate the limits to the two sequences.

3. Show that the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are adjacent. Are they convergent.

Correction

1.$\forall n\in\mathbb{N}\ :\ a_{n+1}\times b_{n+1}=\dfrac{a_n+b_n}{2}\times\dfrac{2\ a_nb_n}{a_n+b_n}=a_n\times b_n=\ldots=a_0\times b_0$ so the sequence $(a_n\times b_n)_{n\in\mathbb{N}}$ is stationary.

2. If $\lim\limits_{n\to+\infty}(a_n)=a$ *and* $\lim\limits_{n\to+\infty}(b_n)=b$ (assuming they converge) then

$$a_{n+1}=\frac{a_n+b_n}{2}\implies a=\frac{a+b}{2}\implies a-b=0.$$

On the other hand, $a_n\times b_n=a_0\times b_0\implies a\ b=a_0\ b_0$ we deduce $a=b=\sqrt{a_0\ b_0}$.

3. We want to show that the two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converge. For this we will prove that they are adjacent

$a_{n+1}-a_n=\dfrac{a_n+b_n}{2}-a_n=\dfrac{b_n-a_n}{2}<0$ so, the sequence $(a_n)_{n\in\mathbb{N}}$ is decreasing.

$b_{n+1}-b_n=\dfrac{2\ a_nb_n}{a_n+b_n}-b_n=\dfrac{(a_n-b_n)\,b_n}{2}>0$, we deduce that the sequence $(b_n)_{n\in\mathbb{N}}$ is increasing.

According to the previous question obviously $\lim\limits_{n\to+\infty}(a_n-b_n)=0$.

So $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are then adjacent and consequently they both converge to the common limit $\sqrt{a_0\ b_0}$.

Notice : For $a,b>0$ we have different averages

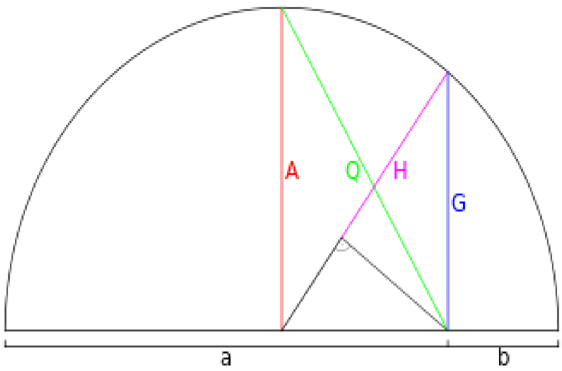
$$Q=\sqrt{\frac{a^2+b^2}{2}}\qquad A=\frac{a+b}{2}\qquad G=\sqrt{a\,b}\qquad H=\frac{2}{\frac{1}{a}+\frac{1}{b}}=\frac{2\ a\,b}{a+b}.$$

The harmonic mean is denoted by H , the arithmetic mean is A and the geometric mean is G .

Q denotes a fourth mean, the quadratic mean.

Since a hypotenuse is always longer than a leg of a right triangle, the diagram shows that

Q > A > G > H.



***Exercise (**Arithmetic mean geometric)*

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences such as $a_0>b_0>0$ and $\forall n\in\mathbb{N}$

$$a_{n+1}=\frac{a_n+b_n}{2}\quad\text{and}\quad b_{n+1}=\sqrt{a_n\,b_n}$$

Show that the two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converge to the same limit.

3 EXAMPLES OF REMARKABLE SUITES

3.1. Approximation of reals by decimals

Proposition

$a \in \mathbb{R}$, posing $u_n = \dfrac{E(10^n\, a)}{10^n}$, *$u_n$ is a* **decimal approximation** *of a within 10^{-n}, in particular* $\lim\limits_{n \to +\infty} u_n = a$.

Indeed: according to the definition of the integer part, we have

$$E(10^n\, a) \leq 10^n\, a < E(10^n\, a) + 1,$$

so $u_n \leq a < u_n + \dfrac{1}{10^n}$ **i.e.** $0 \leq a - u_n < \left(\dfrac{1}{10}\right)^n \xrightarrow[n \to +\infty]{} 0.$

Notice:

1. The terms u_n are decimal numbers, in particular they are rational numbers.

2. So we have for every $a \in \mathbb{R}$, there is a sequence of rational numbers $(u_n)_{n\in\mathbb{N}}$ which converges to a. We say that the rational set $\mathbb{Q}$ is **dense** in the set of real numbers $\mathbb{R}$.

3.2. Linear recurrent sequence of order 1

Definition (Arithmetic sequence)

A sequence $(u_n)_{n\in\mathbb{N}}$ is said to be **arithmetic** *of ratio $(r \in \mathbb{R})$ and first term u_0 if*

$$\forall n \in \mathbb{N} : u_{n+1} = u_n + r.$$

We notice that $\forall n \in \mathbb{N} : u_n = u_0 + n\, r.$

EXAMPLE

1)Constant sequences are arithmetic sequences with ratio $r = 0$.

2) The set $\mathbb{N}$ constitutes the set-image (range) of an arithmetic sequence of first term $u_0 = 0$ and reason $r = 1$.

Exercise

The sale price of a car initially marketed in 1995 decreases every year by the same value. In 2002, it was displayed at a price of €13,200. In 2006, we note a sale price of

€11,600. We note $(u_n)_{n\in\mathbb{N}}$ the selling price of this model in the year (1995+n) and we consider the sequence $(u_n)_{n\in\mathbb{N}}$.

1. Give the nature of $(u_n)_{n\in\mathbb{N}}$.
2. What was the initial sale price in 1995?
3. From what year will it be possible to acquire the car for less than €10,000?
4. From the beginning of 1999 to the end of 2010, a dealer buys ten of these models every year. Determine the total amount spent to purchase all of these vehicles.

Correction

1) The selling price decreases every year by the same value then the sequence $(u_n)_{n\in\mathbb{N}}$ is arithmetic

$$u_{n+1} = u_n + r = u_0 + n\,r.$$

We have $u_7 = u_0 + 7r = 13200$ and $u_{11} = u_0 + 11r = 11600$ from where
$u_7 - u_{11} = -4r = 1600,$
we deduce the ratio from the sequence is $r = -400$.
2) The initial sale price is $u_0 = u_7 - 7r = 13200 - 7(-400) = 16000.$
3)$u_{n+1} = u_0 + n\,r < 10000 \iff 16000 - 400\,n < 10000 \iff n > \dfrac{6000}{400} = 15;$
so it is from the year 2010 that it will be possible to acquire the car for less than 10000€.
4) The total amount spent from 1999 to the end of 2010 is

$$S = u_4 + ... + u_{15} = 12\frac{u_4 + u_{15}}{2} = 11\frac{2\times u_0 + 4r + 15r}{2} = 12\frac{2\times 16000 - 19\times 400}{2} = 146400$$

Definition (Geometric sequence)

A sequence $(u_n)_{n\in\mathbb{N}}$ is said to be **geometric** *of ratio $(q \in \mathbb{R})$ and first term u_0 if*

$$\forall n \in \mathbb{N} : u_{n+1} = q \times u_n.$$

We notice that $\forall n \in \mathbb{N} : u_n = u_0\, q^n.$

Example

1) Constant sequences are geometric sequences with ratio $q = 1$.

2) Let $(u_n)_{n\in\mathbb{N}}$ be the geometric sequence defined by: $u_0 = 1$ and ratio $q = 2$, then $\forall n \in \mathbb{N} : u_n = 2^n$, **i.e.** $(u_n)_{n\in\mathbb{N}} = (1\ ,\ 2\ ,\ 4\ ,\ 8\ ,\ 16\ ,\ 32\ ,\ ...$

Exercise

A microbial population sees its number increase by about 10% every hour. Knowing that it has 200 individuals when we observe it, what will happen after 24 hours?

Correction

Note v_n the number of individuals after n hours. The population increases by 10% every hour so

$$v_{n+1} = v_n + v_n \times 0.1 = v_n\,(1+0.1) = v_0\,(1+0.1)^n,$$

where $v_0 = 200$ *is the* initial population. The sequence $(v_n)_{n\in\mathbb{N}}$ is geometric with ratio $q = 1,1.$

After 24 hours, there will be approximately $v_{24} = 200 \times (1,1)^{24} = 1970$ people.

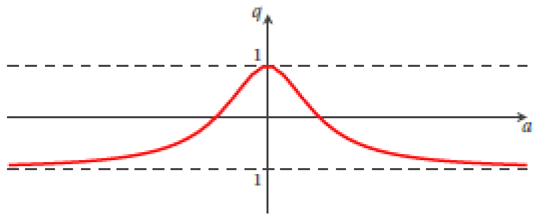
Exercise

Determine the nature of the following sequence and study its convergence:

$$u_n = \left(\frac{1-a^2}{1+a^2}\right)^n.$$

Correction

It is a geometric sequence with common ratio $q(a) = \left(\frac{1-a^2}{1+a^2}\right)$. We study and represent the function $a \to q(a)$



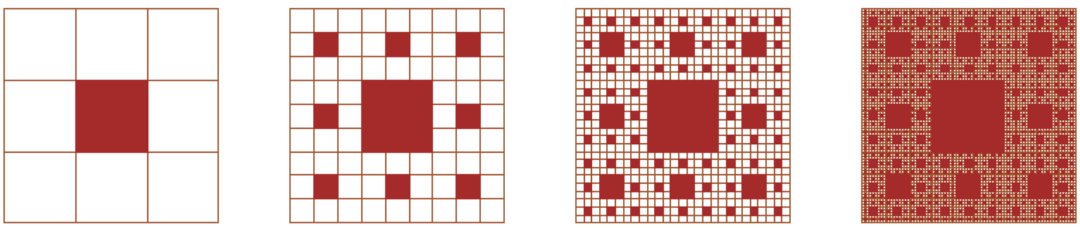
So (u_n) *converge* $\iff$ $-1 < q(a) < +1$, we deduce from the curve that (u_n) *converge* $\forall a \in \mathbb{R}^*$ and in this case $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \big(q(a)\big)^n = 0.$

For $a = 0$ we will have$q(0) = 1$ and then $u_n = 1$, $\forall n \in \mathbb{N}$, we'll have $\lim_{n\to+\infty} u_n = 1.$

Exercise

A unit square is divided into 9 identical squares, the central square being coloured (*Step 1*). Each of the remaining eight squares is divided according to the same principle, and we repeat this process ad infinitum. What will be the area of the coloured surface?

Correction



Note s_n the area of the colored surface in step n , $n \in \mathbb{N}^*$; we have $s_1 = \frac{1}{9}$. To calculate the area s_{n+1} of the colored domain at the step $n+1$, just add to s_n one-ninth of the area of the remaining surface namely $1 - s_n$. We then obtain the following relationship:

$$s_{n+1} = s_n + \frac{1-s_n}{9} = \frac{8}{9}\,s_n + \frac{1}{9}.$$

It is an arithmetic-geometric sequence; therefore, at limit, the area of the coloured domain is equal to $\lim\limits_{n\to+\infty} s_n = l$.

l is the solution of the equation $l = \frac{8}{9} l + \frac{1}{9}$ which yields to $l = 1$.

3.3. Linear recurrent sequence of order 2

Definition (Linear recurrent sequence of order 2)

A linear recurrent sequence of order 2 is given by

$$u_0 = A \ , \ u_1 = B \quad ; \qquad u_{n+2} = a\, u_{n+1} + b\, u_n - - - (Eq)$$

where A, B, a and b fixed real numbers.

Let us look for sequences of the geometric type satisfying this system. The general term of this sequence can be of form $u_n = r^n$ with

$$r^{n+2} = a\, r^{n+1} + b\, r^n \qquad \text{i.e.} \quad r^2 - a\, r - b = 0.$$

If $\Delta = a^2 + 4b > 0$, then there are two distinct real roots:

$$r_1 = \frac{a - \sqrt{a^2 + 4b}}{2} \qquad \text{and} \quad r_= \frac{a + \sqrt{a^2 + 4b}}{2};$$

thus, any solution of (Eq) is of the form

$$u_n = \alpha r_1^n + \beta r_2^n.$$

Now we just have to find the coefficients α and β. Considering the "initial" conditions $u_0 = A \ , \ u_1 = B$ we have to solve the system

$$\begin{cases} \alpha + \beta = A \\ \alpha r_1 + \beta r_2 = B \ . \end{cases}$$

Exercise: *(SCONTINUATION OFFIBONACCI)*

The sequence of FIBONACCI is a sequence of integers such that *each term is the sum of the two precedent terms*. It usually starts with the terms 0 and 1 (sometimes 1 and 1).

It gets its name from Leonardo FIBONACCI, an Italian mathematician from XIIIe century who described the growth of a rabbit colony:

"A man puts a couple of rabbits in an isolated place. How many couples do we get in a year if each couple generates a new couple every month from the third month of its existence?"

In this (ideal) population, we assume that:

1)at (beginning of) the first month, there is just one pair of young rabbits;

2)the rabbits do not procreate until (beginning of) the third month;

3) each (beginning of) month, any pair likely to procreate effectively generates a new pair of young rabbits;

4) there is no mortality (hence the FIBONACCI sequence is strictly increasing).

Correction

Note F_n the number of pairs of rabbits at the beginning of month n. We have by hypothesis $F_1 = F_2 = 1$ and $F_3 = 3$.

In the month $n+2$ *we will have* F_{n+2} pairs ; it results by adding F_{n+1} couples of the month $n+1$ and the newborns that correspond to rabbits aged at least two months, that is F_n couples.

We deduce from this analysis

$$F_{n+2} = F_{n+1} + F_n.$$

We put $F_0 = 0$, we thus obtain the recurrent form of the FIBONACCI sequence: each term of this sequence is the sum of the two previous terms:

$$F_0 = 0\ ,\ F_1 = 1 \quad ; \qquad F_{n+2} = F_{n+1} + F_n --- (Eq).$$

Looking for solutions in the form $u_n = r^n$, the polynomial characteristic is $r^2 - r - 1 = 0$, $\Delta = 5$ and roots will be

$$r_1 = \frac{1-\sqrt{5}}{2}, \qquad\qquad r_= \frac{1+\sqrt{5}}{2} = 1 - r_1.$$

Thus, any solution of (Eq) is of the form (called BINET formula):

$$F_n = \alpha r_1^n + \beta(1-r_1)^n.$$

Considering the "initial" conditions we get the system

$$\begin{cases} \alpha + \beta = 0 \\ \alpha r_1 + \beta(1-r_1) = 1\ . \end{cases} \iff \alpha = -\beta = \frac{1}{\sqrt{5}}$$

hence the number of couples in the n^{eme} year $\qquad F_n = \frac{1}{\sqrt{5}}\big(r_1^n - (1-r_1)^n\big).$

3.4. Recursive sequence defined by a function

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. A ***recurring sequence*** is defined by its first term and a relation allowing to calculate the terms step by step (*successively*):

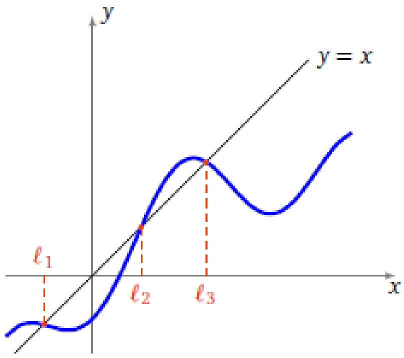
$$u_0 \in \mathbb{R} \quad et \quad u_{n+1} = f(u_n) \quad pour\ n \geq 0.$$

A given recurrent sequence is not necessarily convergent. When the limit exists, the set of possible values is restricted by the following result.

Proposition:

*If the recurring sequence $(u_n)_n$ converges to $l \in \mathbb{R}$ and if the function f is a continuous, **then** l **is a solution of the equation**:* $\quad f(x) = x.$

So, if the limit exists, this proposition affirms that it is to be found among the solutions of the equation $f(l) = l$ (*fixed point of the function* f).



Definition (Fixed point):
A value $l \in \mathbb{R}$, checking $f(l) = l$ is called ***fixed point*** *of the function f.*

We are going to study in detail two particular cases: increasing and decreasing functions.

3.4.1. Case of an increasing function

For an increasing function, the behaviour of the sequence $(u_n)_n$ defined by recurrence $u_{n+1} = f(u_n)$ is quite simple:

Case 1: $u_1 \geq u_0 \implies u_2 = f(u_1) \geq u_1 = f(u_0) \implies \ldots \implies (u_n)_n$ **increasing sequence.**

Case 2: $u_1 \leq u_0 \implies u_2 = f(u_1) \leq u_1 = f(u_0) \implies \ldots \implies (u_n)_n$ **decreasing sequence.**

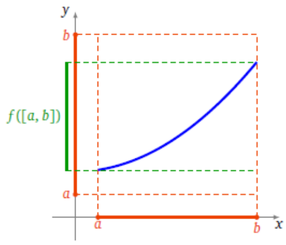
Here is the main result:

Proposition

If $f : [a,b] \to [a,b]$ is a continuous and increasing function, then $\forall\, u_0 \in [a,b]$, the recurring sequence defined by $u_{n+1} = f(u_n)$ is monotonic and converges to $l \in [a,b]$ checking the equation $f(l) = l$.

Notice : There is an important assumption that is somewhat hidden:

$$f\big([a,b]\big) \subset [a,b].$$



Exercise

Either f the function defined by $f(x) = \frac{1}{4}(x^2 - 1)(x - 2) + x$.
Study the recurrent sequence defined by $u_n \in [0,2]$ and $u_{n+1} = f(u_n)\ for\ n \geq 0$.

Correction

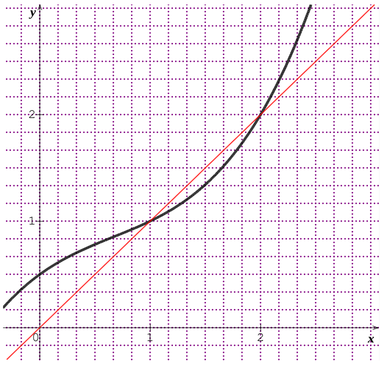
1.Study of the function f:

(a) f is continuous on R.

(b) f is differentiable on R, f'(x) > 0 on the interval [0, 2] so that f is strictly increasing.

(d) f (0) = 1/2 and f (2) = 2 then $f([0,2]) = [1/2, 2] \subset [0,2]$.

We conclude that the sequence is bounded.



2. Calculation of fixed points, values that satisfy $f(x) = x$:
For this let's solve the equation $f(x) - x = \frac{1}{4}(x^2 - 1)(x - 2) = 0$.
Fixed points are $-1, +1, 2$. The limit of u_n is to be found among the values $+1, 2$.

3. Convergence of the recurrent sequence (Separation of cases):
From the representative curve C_f intersect the prime bisector in points of abscissa +1 and +2. For $u_0 = 1$ or $u_0 = 2$. the sequence will be constant: $u_n = u_0 \; \forall n > 0$.
Otherwise, we have

Case 1: $0 \leq u_0 < 1$; *in this case* $(f(x) > x)$ $u_1 \geq u_0$ *then the sequence.* $(u_n)_n$ *is increasing and upper-bound by 2, therefore it converges to the fixed point* $l = +1$.

Case 2: $1 < u_0 < 2$;*in this case* $(f(x) < x)$ $u_1 \leq u_0$ *then the sequence.* $(u_n)_n$ *is decreasing and lower-bound by 0, so it converges to the fixed point* $l = +1$.

Notice:

The graph of the function plays a very important role, it must be drawn even if it is not explicitly requested. It allows you to get a very precise idea of the behavior of the sequence.

3.4.2. Case of a decreasing function

If the function f is decreasing then $f \circ f$ is increasing, applying the previous result to $f \circ f$ we obtain:

Proposition

If $f : [a,b] \to [a,b]$ *a continuous function and* ***decreasing****, then* $\forall u_0 \in [a,b]$, *the sequence defined by* $u_{n+1} = f(u_n)$ *check what follows:*

- *The sub-sequence* $(u_{2n})_n$ *converges to a limit* $l \in [a,b]$ *checking* $f \circ f(l) = l$.
- *The sub-sequence* $(u_{2n+1})_n$ *converges to a limit* $l' \in [a,b]$ *checking* $f \circ f(l') = l'$.
- *It may (or not)* $l = l'$.

Exercise

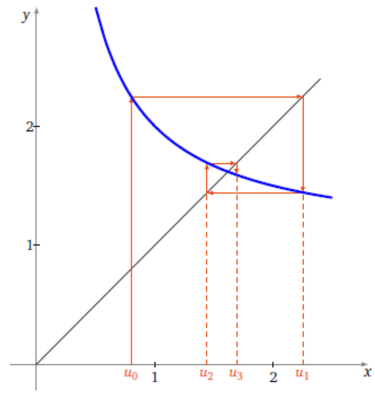
Let f be the function defined by $f(x) = 1 + \dfrac{1}{x}$. Study the recurrent sequence defined by

$$u_0 > 0, \qquad u_{n+1} = f(u_n) \qquad \text{for } n \geq 0.$$

Correction

1. Study of f

The function f is continuous and strictly decreasing on $]0,+\infty[$. $f(]0,+\infty[) =]1,+\infty[\subset]0,+\infty[$ then $f \circ f(]0,+\infty[) \subset]0,+\infty[$.



2 Calculation of fixed points.

Find the x values satisfying $f \circ f(x) = x$.

$$f \circ f(x) = 1 + \frac{1}{1 + \frac{1}{x}} = 1 + \frac{x}{x+1} = \frac{2x+1}{x+1}.$$

$$f \circ f(x) = x \iff \frac{2x+1}{x+1} = x \iff x^2 - x - 1 = 0 \iff x \in \left\{\tfrac{1-\sqrt{5}}{2}, \tfrac{1+\sqrt{5}}{2}\right\}.$$

$\frac{1-\sqrt{5}}{2} < 0$ so, the only fixed point to consider is $\frac{1+\sqrt{5}}{2}$.

Note that the sign of $f \circ f(x) - x = \dfrac{2x+1}{x+1} - x = \dfrac{-x^2+x+1}{x+1}$ informs us that:

(*) $0 < x \leq \frac{1+\sqrt{5}}{2} \implies f \circ f(x) \geq x$ and (**) $x \geq \frac{1+\sqrt{5}}{2} \implies f \circ f(x) \leq x$

3. Convergence of the recurrent sequence (Separation of cases):

Case: 1 $0 < u_0 \leq \dfrac{1+\sqrt{5}}{2}$;

1) From (*) $0 < u_0 \leq \frac{1+\sqrt{5}}{2} \implies u_2 = f \circ f(u_0) \geq u_0$, $f \circ f$ being increasing, then the sequence $(u_{2n})_n$ is increasing.

2) From (**) $u_1 \geq \frac{1+\sqrt{5}}{2} \implies u_3 = f \circ f(u_1) \leq u_1$ $f \circ f$ being increasing, then the sequence $(u_{2n+1})_n$ is decreasing.

3) f is creasing then $0 < u_0 \leq \dfrac{1+\sqrt{5}}{2} \implies u_1 = f(u_0) \geq f\left(\dfrac{1+\sqrt{5}}{2}\right) = \dfrac{1+\sqrt{5}}{2} \geq u_0$; *we deduce that*

$$u_0 \leq u_2 \leq u_4 \leq \ldots \leq u_{2n} \leq \ldots \ldots \leq u_{2n+1} \leq \ldots \leq u_5 \leq u_3 \leq u_1.$$

__We conclude:__

1) the sequence $(u_{2n})_n$ *is increasing and upper-bounded by* u_1, *so it converges to the unique fixed point* $l = \frac{1+\sqrt{5}}{2}$.

2) the sequence $(u_{2n+1})_n$ *is decreasing and lower-bounded by* u_0, *so it converges to the unique fixed point* $l = \frac{1+\sqrt{5}}{2}$.

So, the sequence $(u_n)_n$ *converges to* $l = \frac{1+\sqrt{5}}{2}$.

Case 2: *we have* $\quad u_0 \geq \dfrac{1+\sqrt{5}}{2} \implies u_1 = f(u_0) \leq f\left(\dfrac{1+\sqrt{5}}{2}\right) = \dfrac{1+\sqrt{5}}{2} \leq u_0$;

we deduce that:

1) From (*) $\quad u_0 \geq \frac{1+\sqrt{5}}{2} \implies u_2 = f\circ f(u_0) \leq u_0$, $\ f\circ f$ being increasing, then the sequence $(u_{2n})_n$ is decreasing.

2) From (**) $\quad u_1 \leq \frac{1+\sqrt{5}}{2} \implies u_3 = f\circ f(u_1) \geq u_1$, $\ f\circ f$ being increasing, then the sequence $(u_{2n+1})_n$ is increasing.

3) f *is* creasing then $\qquad u_0 \geq \dfrac{1+\sqrt{5}}{2} \implies u_1 = f(u_0) \leq f\left(\dfrac{1+\sqrt{5}}{2}\right) = \dfrac{1+\sqrt{5}}{2} \leq u_0$;

we deduce that

$$u_1 \leq u_3 \leq u_5 \leq \ldots \leq u_{2n+1} \leq \ldots \ldots \leq u_{2n} \leq \ldots \leq u_4 \leq u_2 \leq u_0.$$

We conclude:

1) the sequence $(u_{2n})_n$ *is decreasing and lower-bounded by* u_1, *so it converges to the unique fixed point* $l = \frac{1+\sqrt{5}}{2}$.

2) the sequence $(u_{2n+1})_n$ *is increasing and upper-bounded by* u_0, *so it converges to the unique fixed point* $l = \frac{1+\sqrt{5}}{2}$.

We deduce that the sequence $(u_n)_n$ *converges to* $l = \frac{1+\sqrt{5}}{2}$.

Exercise

Study the sequence defined by u_0 , $u_{n+1} = f(u_n)$ $n \in \mathbb{N}$ for the following cases:

1) $u_0 > 1, f(x) = 2 - \dfrac{1}{x}$, $\quad$ **2)** $1 \leq u_0 < 2, f(x) = \sqrt{2+x}$, $\quad$ **3)** $u_0 > 2, f(x) = x^2+1$,

4) $u_0 > 0, u_{n+1} = \dfrac{u_n}{1+u_n^2}$, $\quad$ **5)** $0 < u_0 < 1, u_{n+1} = \dfrac{u_n^2+u_n}{2}$.

Correction

1) $u_0 > 1, u_{n+1} = f(u_n) = 2 - \dfrac{1}{u_n}$. $x \to f(x)$ is increasing.

a) Let us show that $u_n > 1$, $\forall n \in \mathbb{N}$: we have $u_0 > 1$, suppose $u_n > 1$ then

$u_{n+1} \geq 2 - \dfrac{1}{1} = 1$;

hence by the recurrence principle we get: $\forall n \in \mathbb{N}$: $u_n > 1$.

b) Let us study the variation of (u_n) :

$$u_{n+1} - u_n = 2 - \frac{1}{u_n} - u_n = \frac{2u_n - 1 - u_n^2}{u_n} = \frac{-(u_n-1)^2}{u_n} < 0$$

so (u_n) is decreasing.

c) Let's show that (u_n) is convergent: $\qquad (u_n)$ is decreasing and *lower-bounded* by 1 so it converges.

d) seek the limit: note its limit l, we must have $l \geq 1$.

Seeking for the fixed points: $u_{n+1} = 2 - \dfrac{1}{u_n} \implies l = 2 - \dfrac{1}{l}$, so l is solution of equation $l^2 - 2l + 1 = 0$, it is $l = 1$. It is the limit of (u_n).

2) $1 \leq u_0 < 2, u_{n+1} = f(u_n) = \sqrt{2 + u_n}$, $x \to f(x)$ is increasing.

a) Let us show that $1 \leq u_n < 2$, $\forall n \in \mathbb{N}$: we have $1 \leq u_0 < 2$, suppose $1 \leq u_n < 2$, then

$$1 \leq \sqrt{3} = \sqrt{2+1} \leq u_{n+1} = \sqrt{2 + u_n} < \sqrt{2+2} = 2,$$

hence by the principle of recurrence we get $\forall n \in \mathbb{N} \ : \ 1 \leq u_n < 2$.

b) Let us study the variation of (u_n) : $u_{n+1} - u_n = \sqrt{2 + u_n} - u_n = \dfrac{2 + u_n - u_n^2}{\sqrt{2 + u_n} + u_n}$ the denominator is positive, let's study the sign of the numerator: $-x^2 + x + 2, \Delta = 9, x_1 = 2$, $x_2 = -1$, for $1 \leq x < 2$ we have $-x^2 + x + 2 > 0$, we deduce that (u_n) is increasing.

c) Show that (u_n) is convergent: (u_n) is increasing and *upper-bounded* by 2, so it converges.

d) seek the limit: denote by l the limit, we must have $1 \leq l \leq 2$.

Seeking for the fixed points: $u_{n+1} = \sqrt{2 + u_n} \implies l = \sqrt{2 + l}$, l is solution of equation $l^2 - l - 2 = 0$, $\Delta = 9$, solutions $l_1 = 2$ and $l_2 = -1$. As $1 \leq u_n < 2$ and (u_n) is increasing, we conclude the limit is $l = 2$.

3) $u_0 > 2$, $u_{n+1} = f(u_n) = u_n^2 + 1$, $x \to f(x)$ is increasing on $]2, +\infty[$.

a) Let us show that $u_n > 2$, $\forall n \in \mathbb{N}$: we have $u_0 > 2$, suppose $u_n > 2$, then

$$u_{n+1} = u_n^2 + 1 > 2^2 + 1 = 5 > 2,$$

hence by the recurrence principle $\forall n \in \mathbb{N} \ : \ u_n > 2$.

b) Let us study the variation of (u_n) :

$u_{n+1} - u_n = u_n^2 + 1 - u_n \geq u_n^2 + 1 - 2u_n = (u_n - 1)^2 > 0$ on $]2, +\infty[$, so (u_n) is increasing.

c) Convergence: (u_n) is increasing and *lower-bounded* by 2; we cannot conclude anything on its convergence.

If it converges let us denote by l its limit, we must have $l \geq 2$ and $u_{n+1} = u_n^2 + 1 \implies l = l^2 + 1$; l is solution of $l^2 - l + 1 = 0, \Delta = -3$. There is no real solution and therefore (u_n) cannot be convergent.

As (u_n) is positive strictly increasing then it diverges towards $+\infty$.

4) $u_0 > 0, u_{n+1} = \dfrac{u_n}{1 + u_n^2}$. Consider the function $x \to f(x) = \dfrac{x}{1 + x}$. We have

$$f'(x) = \frac{1(1+x) - x(1)}{(1+x)^2} = \frac{1}{(1+x)^2} > 0,$$ so the function is increasing.

a) Let us show that $u_n > 0$, $\forall n \in \mathbb{N}$: we have $u_0 > 0$, suppose $u_n > 0$, then $u_{n+1} = f(u_n) > f(0) = 0$,

hence by the recurrence principle $\forall n \in \mathbb{N} \ : \ u_n > 0$.

b) Let us study the variation of (u_n): $u_{n+1} - u_n = \dfrac{u_n}{1 + u_n^2} - u_n = \dfrac{-u_n^3}{1 + u_n^2} < 0$, (u_n) is decreasing.

c) Convergence: (u_n) is decreasing and ***lower-bounded*** by 0, so it converges. Note l its limit, we must have $l \geq 0$ and $l = \dfrac{l}{1+l^2}$; so l is solution of equation $l^3 = 0$ that is $l = 0$.

5) $0 < u_0 < 1, u_{n+1} = \dfrac{u_n^2 + u_n}{2}$. Consider the function $x \to f(x) = \dfrac{x^2 + x}{2}$; we have $f'(x) = x + \dfrac{1}{2} > 0$ so the function is increasing on $]0,1[$.

a) Let us show that $0 < u_n < 1$, $\forall n \in \mathbb{N}$: we have $0 < u_0 < 1$, suppose $0 < u_n < 1$, then $0 = f(0) < u_{n+1} = f(u_n) < f(1) = 1$, hence by the recurrence principle

$$\forall n \in \mathbb{N} \ : \ 0 < u_n < 1.$$

b) Let us study the variation of (u_n):

$$u_{n+1} - u_n = \frac{u_n^2 + u_n}{2} - u_n = \frac{u_n^2 - u_n}{2} = \frac{u_n\,(u_n - 1)}{2} < 0$$

so (u_n) is decreasing.

c) Convergence: (u_n) is decreasing and ***lower-bounded*** by 0, so it converges.

Note by l its limit, we must have $0 \leq l \leq 1$ and $l = \dfrac{l^2 + l}{2}$, so l is solution of equation $l^2 - l = 0$, that is $l = 0 \vee l = 1$. As $0 < u_n < 1$ and (u_n) is decreasing, we deduce that $l = 0$.

3rd chap. : Generalities on Functions

1 Generalities

1.1. Definition, graph, equality, restriction, intersections

Definition :

*A **function** $f\ :\ D_f \subset \mathbb{R} \to \mathbb{R}$, is a "process" which at each real $x \in D_f$ (input) associates (at most) one real number $y \in \mathbb{R}$ (image) noted $f(x)$.*

We notice:

$$f\ :\ E \subset D_f \to \mathbb{R}$$
$$x \to y = f(x)\ .$$

*The **domain** D_f is the greatest set of real numbers x for which $f(x)$ exists.*

Notice

You can define a function in different ways:

1) using an expression such as: $f(x) = \dfrac{1}{x+1}$ with $D_f = \mathbb{R} \setminus \{-1\}$;

2) using several expressions: $g(x) = \begin{cases} -x & \text{si } x \leq 0 \\ \sin x & \text{sinon} \end{cases}$ with $D_g = \mathbb{R}$;

3) using certain curves, for example an electrocardiogram.

Notice

1) Attention: do not confuse the function f and the real f(x).

2) The variable x is mute; we can very well write $t \to f(t)$ Or $\blacksquare \to f(\blacksquare)$.

Exercise

Let be the function defined by $f(x) = \dfrac{1}{x-3}$

Give $f(4), f(3), 4f(x), f(4x), f(x+4), f(4)+f(x), f(-x), -f(x)$.

Correction

$f(4) = \dfrac{1}{4-3} = 1$, $f(3)$ doesn't exist, $4f(x) = \dfrac{4}{x-3}$, $f(4x) = \dfrac{1}{4x-3}$,

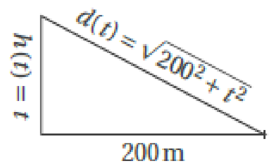
$f(x+4) = \dfrac{1}{(x+4)-3} = \dfrac{1}{x+1}$, $f(4)+f(x) = \dfrac{1}{4-3} + \dfrac{1}{x-3} = 1 + \dfrac{1}{x-3} = \dfrac{x-2}{x-3}$,

$f(-x) = \dfrac{1}{-x-3} = \dfrac{-1}{x+3}$, $-f(x) = \dfrac{-1}{x-3}$.

Exercise

A hot air balloon rises vertically from the ground at a speed of $1\ m/s$.
Express, as a function of height, the distance between the balloon and an observer initially located 200 m away (see drawing below).

Correction



We apply the Pythagorean theorem:

$$d^2 = h^2 + 200^2 = (v.t)^2 + 200^2 \quad \text{then} \quad d(t) = \sqrt{t^2 + 200^2}$$

Exercise

We want to build a steel tank for propane gas in the shape of a cylinder of 10 m long with a hemisphere at each end (see picture).
Express the volume of the tank (in m^3) as a function of the radius r (in m).

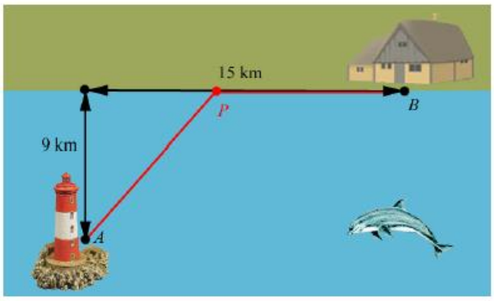
Correction

V(total) = V(half-ball) + V(cylinder) + V(half-ball) = V(ball) + V(cylinder).

$$V(total) = \frac{4}{3}\pi r^3 + \pi r^2 h = \frac{4\pi}{3} r^3 + 10\pi\, r^2.$$

Exercise

The keeper of a lighthouse (point A) must join his house (point B). He travels by canoe at a speed of 4 km/h and on foot at a speed of 5 km/h. The coast is assumed to be straight. It will dock at point P such that$PB = x$. If t is the total time to reach home, express t in terms of x.



Correction

At constant speed v the distance traveled in one time t East $x = v\,t$. $x = PB$ So, using Pythagoras theorem

$$AP = \sqrt{9^2 + (15 - x)^2}.$$

To go from A to P the goalkeeper will put

$$t = \frac{AP}{v} + \frac{PB}{v} = \frac{\sqrt{9^2 + (15 - x)^2}}{4} + \frac{x}{5} = \frac{\sqrt{x^2 - 30x + 306}}{4} + \frac{x}{5}.$$

Notice

Usually, the **domain** D_f of a function is not given, it needs to be specified.

Exercise

Decide if the relationships below are functions of x. If yes, find the definition domain D

1) $y = (x+2)^2$ 2) $y = \dfrac{1}{(x+2)^2}$ 3) $y = \dfrac{1}{x^2+2}$ 4)$y = \pm 3x$

5) $y = \dfrac{1}{\sqrt{x^2+2x+1}}$ 6) $y^2 = x^2$ 7) $y = \dfrac{|x|}{x}$ 8) $y = \sqrt{2-x}$.

Correction

1) $y = (x+2)^2$ $D = \mathbb{R}$ 2) $y = \dfrac{1}{(x+2)^2}$ $D = \mathbb{R} \setminus \{-2\}$

3) $y = \dfrac{1}{x^2+2}$ $D = \mathbb{R}$ 4) $y = \pm 3x$ is not a function because every $x \neq 0$ admits two images.

5) $y = \dfrac{1}{\sqrt{x^2+2x+1}} = \dfrac{1}{\sqrt{(x+1)^2}}$ $D = \mathbb{R} \setminus \{-1\}$

6) $y^2 = x^2 \iff y = \pm x$ is not a function because every $x \neq 0$ admits two images.

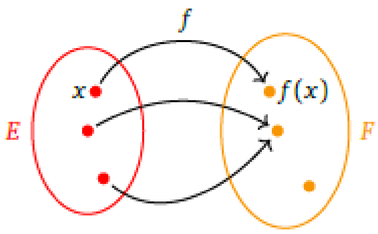
7) $y = \dfrac{|x|}{x}$ $D = \mathbb{R}^*$ 8) $y = \sqrt{2-x}$ $D =]-\infty, 2]$.

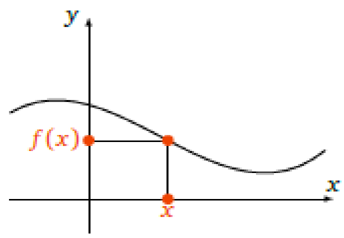
Definition (Graph of a function):

*The **graph** of* $f : D_f \subset \mathbb{R} \to \mathbb{R}$ *is the set* $\Gamma_f = \{(x, f(x)) \in \mathbb{R} \times \mathbb{R} \ , \ x \in D_f\}$.

Usually, we represent Γ_f *by a representative curve noted* C_f

We can represent the functions by two types of illustrations:





Definition (Equality of two functions):

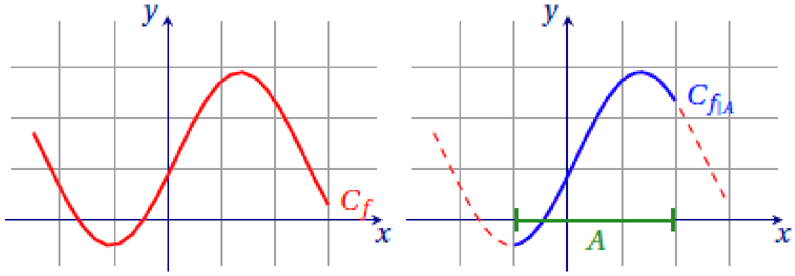
$f, g : E \to F$ *are said to be **equal** if and only if* $\forall x \in E : f(x) = g(x)$.

We denote: $f = g$.

Definition (Restriction of a function):

Let be a function $f : D_f \subset \mathbb{R} \to \mathbb{R}$ *and* $A \subset D_f$*.The function f is well defined on A and we call **restriction** of* f *to A, the function denoted* $f|_A$ *and defined by*

$$f|_A : A \subset D_f \to \mathbb{R}$$
$$x \to y = f|_A(x) = f(x) \ .$$



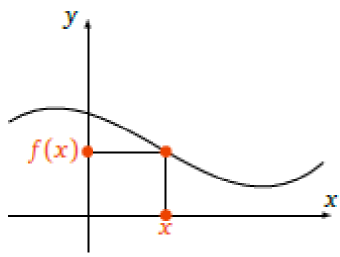
Example

For $g(x)=\begin{cases} -x & \text{si } x\leq 0 \\ \sin x & \text{sinon}\end{cases}$ we have $g|_{]0,+\infty[}(x)=\sin x$ and $g|_{]-\infty,0]}(x)=-x$.

Notice

- ✓ When there is no possible confusion, it happens that we use the notation f to designate $f|_E$. For example, $\cos\ :\ [0,2\pi]\to\mathbb{R}$.
- ✓ We can also restrict the **destination set** to a subset of $\mathbb{R}$.

Recall that the "***graph***" ***(representative curve)*** of $f\ :\ \mathbb{R}\to\mathbb{R}$ is made up of points on the plane whose coordinates are $(x,f(x))\ ,\ x\in\mathbb{R}$.



Intersections:

*Consider $f,g\ :\ \mathbb{R}\to\mathbb{R}$. The **intersection** of curves $C_f\ ,\ C_g$.is given by*

$$C_f\cap C_g=\{(x,f(x))\ :\ f(x)=g(x),\ x\in D_f\cap D_g\}.$$

*Especially for the **intersection with axes** we have:*

- $C_f\cap(x'ox)=\{(x,f(x)\ :\ f(x)=0\ ,\ x\in D_f\}$ *which is the set solution of the equation* $f(x)=0$*;*
- *Whether* $0\in D_f$ *we have* $C_f\cap(y'oy)=\{(0,f(0)\}$.

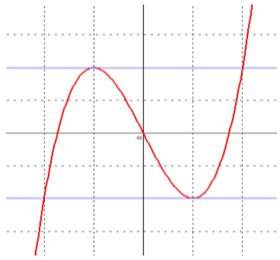
Exercise

Chat graphically according to the value of the parameter $m\in\mathbb{R}$ the number and sign of the solutions of the equation $f(x)=0$ for:

$$1)\ f(x)=x^3-x-m\quad,\quad 2)\ f(x)=\cos(5x)-m.$$

Correction

1) The solutions of the equation $f(x) = 0$ are the abscissas of points of intersection of the curve of $g : x \to x^3 - x$ and the horizontal line with equation $y = m$.



By varying $m \in]-\infty, +\infty[$ and by observing the quoted intersections we obtain:

a) If $m \in]-\infty, -2[$ then the equation $f(x) = 0$ admits a negative solution.

b) If $m = -2$ then the equation $f(x) = 0$ admits a negative solution and a positive solution.

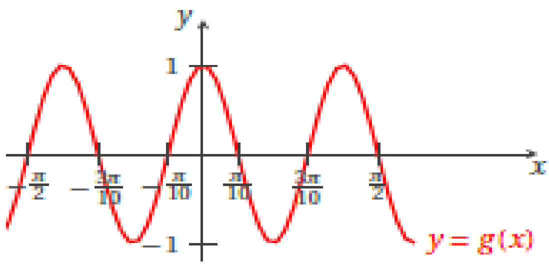
c) If $m \in]-2, 0]$ then the equation $f(x) = 0$ has one negative solution and two positive solutions.

d) If $m \in]0, +2[$ then the equation $f(x) = 0$ admits two negative solutions and one positive solution.

e) If $m = +2$ then the equation $f(x) = 0$ admits a negative solution and a positive solution.

f) If $m \in]2, +\infty[$ then the equation $f(x) = 0$ admits a positive solution.

2) The solutions of the equation $f(x) = 0$ are the abscissas of the points of intersection of the curve of $g : x \to \cos(5x)$ and the horizontal line with equation $y = m$:



a) If $|m| \leq 1$ i.e.,$m \in [-1, -1]$ then the equation $f(x) = 0$ admits an infinity of solutions

b) Otherwise, the equation $f(x) = 0$ does not admit solutions.

1.2. Injection, surjection, bijection

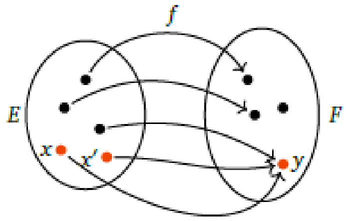
Definition and characterization of an injective function:

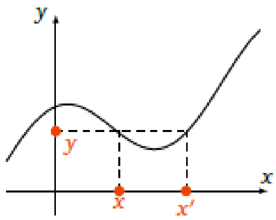
1) **[**$f : E \to F$ *is* **injective]** $\iff x \neq x' \implies f(x) \neq f(x')$

$\iff f(x) = f(x') \implies x = x'$.

2) $f : E \to F$ *is an* **injection** *if and only if* $\forall y \in F$ *the* ***equation*** $y = f(x)$ *admits* ***at most one solution*** $x \in E$.

Below we represent non-injective functions:



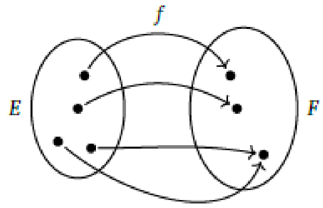


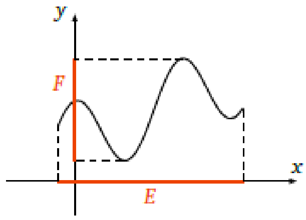
Definition and characterization of a surjective function:

1) **[**$f : E \to F$ *is* **surjective]** $\Longleftrightarrow$ $\forall y \in F$ *il existe* $x \in E$: $y = f(x)$

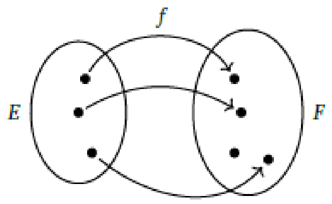
2) $f : E \to F$ *is a* **surjection** *if and only if* $\forall y \in F$ *the* ***equation*** $y = f(x)$ *admits* ***at least one solution*** $x \in E$.

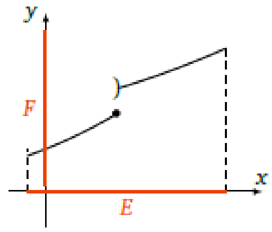
We represent **surjective** and **non-injective** functions





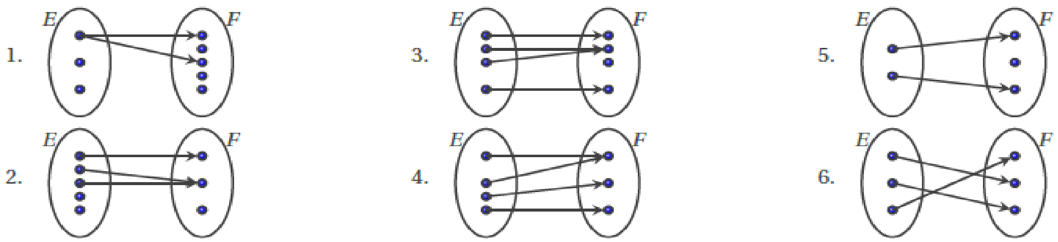
Graphic representation of **injectives** and **non-surjectives** functions.





Exercise

Let E and F be two sets. For every relation $x \mathcal{R} y$ with $x \in E$ and $y \in F$ determine which are functions, then the domain of definition of each of these functions. Determine which are maps and whether they are injective and/or surjective.

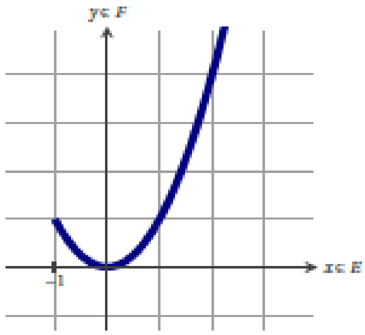


Correction

1	It is not a function.	4	It is a non-injective surjective map.
2	It is a function but it is not an application.	5	It is an injective non-surjective map.
3	It is neither injective nor surjective.	6	It is an injective and surjective map.

Exercise

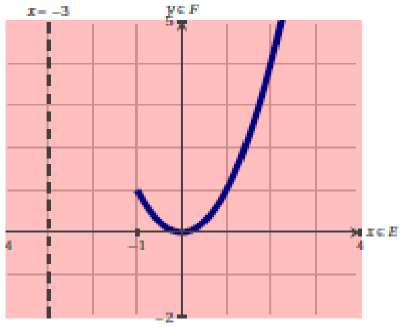
Let E and F be two subsets of $\mathbb{R}$ and a function $f : E \to F$. the graph of which is drawn opposite. For each choice of E and F determine if the function is an application and if it is injective and/or surjective:



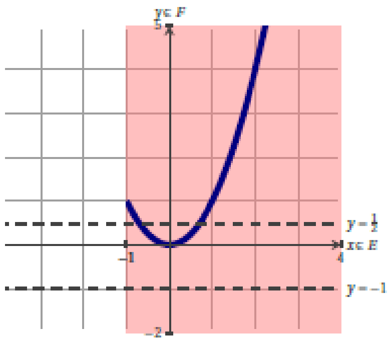
1. $E := \mathbb{R}$ and $F := \mathbb{R}$, **2.** $E := [-1, +\infty[$ and $F := \mathbb{R}$, **3.** $E := [-1, +\infty[$ and $F := [0, +\infty[$, **4.** $E := [-1, 0]$ and $F := [0, +\infty[$, **5.** $E := [-1, 0]$ and $F := [0; 1]$.

Correction

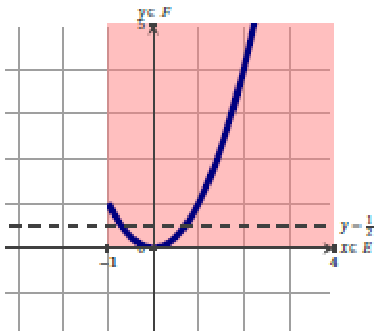
1. $E := \mathbb{R}$ and $F := \mathbb{R}$,:
a) it is a function because any straight line with equation $x = k$, $k \in E := \mathbb{R}$intersects the graph offat most once,
b) it is not an application because the line with equation $x = -3$ never intersects the graph of f.



2. $E := [-1, +\infty[$ and $F := \mathbb{R}$,
a) it is an application because any line with equation $x = k$, $k \in E := [-1, +\infty[$intersects the graph of f exactly once,
b) it is not injective because the line with equation $y = \frac{1}{2}$ intersects the graph of f more than once,
c) it is not surjective because the line with equation $y = -1$ never intersects the graph of the function f.



3. $E := [-1, +\infty[$ and $F := [0, +\infty[$,
a) it is an application because any line with equation $x = k$, $k \in E := [-1, +\infty[$intersects the graph of f exactly once,
b) it is not injective because the line with equation $y = \frac{1}{2}$ intersects the graph of f more than once,
c) it is surjective because any line with equation $y = k$, $k \in E := [0, +\infty[$ intersects the graph of f at least once.

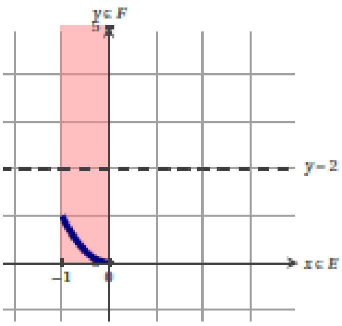


4. $E := [-1, 0]$ and $F := [0, +\infty[$

a) it is an application because any line with equation $x = k$, $k \in E := [-1, 0]$ intersects the graph of f exactly once,

b) it is injective because the line with equation $y = k$, $k \in F := [0, +\infty[$ intersects the graph of f at most once,

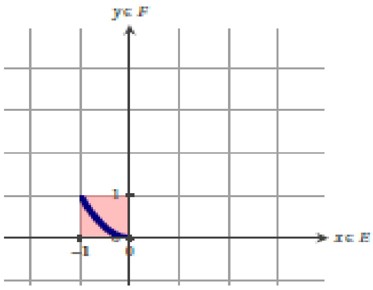
c) it is not surjective because the line with equation $y = 2$ never intersects the graph of f.



5. $E := [-1, 0]$ and $F := [0, 1]$

a) it is an application because any line with equation $x = k$, $k \in E := [-1, 0]$ intersects the graph of f exactly once,

b) it is bijective (injective and surjective) because any straight line with equation $y = k$, $k \in F := [0, 1]$ intersects the graph of f exactly once.

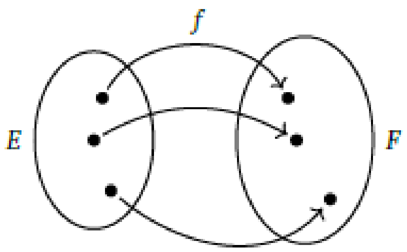


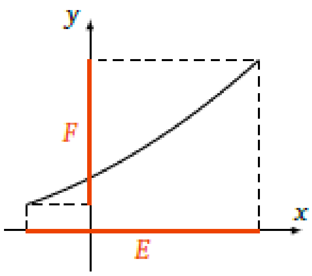
Definition and characterization of a bijective function:

1) *$f : E \to F$ is said to be* **bijective** *if it is an injective and surjective application.*

2) *$f : E \to F$ is a* **bijection** *if and only if $\forall y \in F$ the* ***equation*** *$y = f(x)$ admits* ***a unique solution*** *$x \in E$.*

Below we represent bijective functions:

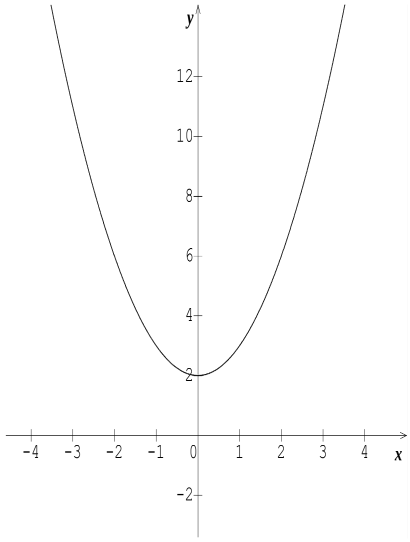




Exercise

Let be the function $f:E\to F$ defined by

$$f(x)=x^2+2.$$



1. suppose that $E=F=\mathbb{R}$.

a) Show that f is a map.

b) Show that f is not injective.

c) Show that f is not surjective.

2. suppose that $E:=]-\infty,0]$ and $F:=[2,+\infty[$.

i) Show that f is a one-to-one map.

ii) Find f^{-1} reciprocal application of f.

Attention: *we can draw inspiration from the graph of f and by considering the intersections of this graph with horizontal lines, but this does not constitute a proof!*

Correction

1.suppose that $E=F=\mathbb{R}$. The curve C_f of f is represented above:

a) $f(x)=x^2+2,\ D_f=\mathbb{R}=E$: the function f is an application.

b)f is injective if $x_1\neq x_2 \implies f(x_1)\neq f(x_2)$; but $f(-1)=f(+1)=3$ so f is not injective.

vs).$f$is surjective if $\forall y\in F=\mathbb{R}$ the equation $y=f(x)$, $x\in E=\mathbb{R}$ admits a solution (at least); but for $y=0$ the equation $0=x^2+2$ does not admit a solution, so$f$is not surjective.

2. suppose that $E=]-\infty,0]$ and $F=[2,+\infty[$.

i) The restriction of$f$to these new sets makes it injective and surjective. Indeed, consider for $y\in F=[2,+\infty[$ the equation$y=f(x)=x^2+2$, $x\in E=]-\infty,0]$:

$$\begin{aligned} y=x^2+2 &\iff x^2=y-2 \\ &\iff |x|=\sqrt{y-2} \quad \left[\text{on a } y\geq 2\right] \\ &\iff x=\sqrt{y-2} \quad \left[\text{on a } x\geq 0\right] \end{aligned}$$

For everything $y\in F$ the equation $y=f(x)$ admits a unique solution $x\in E$, so$f$is bijective.

ii) Reciprocal application $f^{-1}\ :\ F=[2,+\infty[\to E=]-\infty,0]$ is defined by

$$f^{-1}(x)=\sqrt{x-2}.$$

Exercise

Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by $f(x)=\dfrac{2x}{x^2+1}$.

1. Is f injective? surjective? **2.** Show that $f(\mathbb{R})=[-1,1]$.

3. Show that the restriction $g:[-1,1]\to[-1,1]$ with $g(x)=f(x)$ is a bijection.

Correction

Note that f is an application because $D_f = \mathbb{R}$.

1. We have $$y = \frac{2x}{x^2+1} \iff yx^2 - 2x + y = 0$$

the equation of unknown x and parameter $y \neq 0$ admits solutions if and only if the discriminant reduces $\Delta' = 1 - y^2 \geq 0$which s realized for $|y| \leq 1$. We deduce that: has for$y = \frac{1}{2}$ (for example) there will be two solutions and therefore f is not injective.

b) for $y = 3$ (e.g.) there will be no solutions and so f is not surjective.

2. Consider the equation $y = f(x)$ of unknown x and parameter y :

a) for $y \neq 0$ we obtain as solution $x = 0$; **b)** for $y \neq 0$ it admits solutions for$|y| \leq 1$. We deduce that the range of f is $[-1, 1]$.

3. From the results obtained above applied to the restriction $g : [-1,1] \to [-1,1]$, the equation $y = g(x)$ has for all $y \in [-1,1] \setminus \{0\}$ two solutions $x_1 = \frac{1-\sqrt{1-y^2}}{y}$ and $x_2 = \frac{1+\sqrt{1-y^2}}{y}$.

We have $|y| \leq 1$ SO $\frac{1}{|y|} \geq 1$, then $|x_2| = \dfrac{1+\sqrt{1-y^2}}{|y|} \geq \dfrac{1}{|y|} \geq 1$.

We deduce that so only the solution $x_1 = \frac{1-\sqrt{1-y^2}}{y}$is admissible and consequently, the restriction$g : [-1,1] \to [-1,1]$ is a bijection with$g^{-1}(x) = \frac{1-\sqrt{1-x^2}}{x}$.

Exercise

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \ln(|x| + \frac{1}{e})$

1. Is f injective? surjective?
2. Show that the restriction$g : [0,+\infty[\to [-1,+\infty[$ with $g(x) = f(x)$ is a bijection and calculate the reciprocal function g.

Correction

1. Note first that f is a mapping because $D_f = \mathbb{R}$ because $\forall x \in \mathbb{R} \; : \; |x| + \frac{1}{e} \geq \frac{1}{e} > 0$.

a) We have $y = \ln(|x| + \frac{1}{e}) \iff x = \pm(e^y - \frac{1}{e})$. We note that $y = 0$ (for example) have two antecedents and therefore f is not injective.

b) We have$|x| \geq 0 \implies y = \ln(|x| + \frac{1}{e}) \geq \ln(\frac{1}{e}) = -1$. We note that $y = -3$ (for example) has no antecedent and therefore the function f is not surjective.

2. From the above study, we deduce that the restriction $g : [0,+\infty[\to [-1,+\infty[$admits for everything $y \in [-1,+\infty[$ has a single antecedent $x \in [0,+\infty[$.

Consequently $g : [0,+\infty[\to [-1,+\infty[$ is a bijection with reciprocal mapping defined by $g^{-1}(x) = e^x - \frac{1}{e}$.

1.3. Composition of functions, bijection and reciprocal function

Composition of functions:

Let be functions $f\ :\ D_f \to F$ and $g\ :\ F \to \mathbb{R}$. The **composite function** *$g \circ f$ is defined by:* $\quad x \in D_f\,,\ f(x) \in D_g\ :\ (g \circ f)(x) = g\big(f(x)\big).$

$$\begin{tikzcd} \mathscr{D}_f \arrow[r, "f"] \arrow[rr, bend left, dashed, "g \circ f"] & f(\mathscr{D}_f) \arrow[r, "g"] & \mathbb{R} \\ x \arrow[r, mapsto] & f(x) \arrow[r, mapsto] & g(f(x)) \end{tikzcd}$$

Example: $E = F = \mathbb{R}$

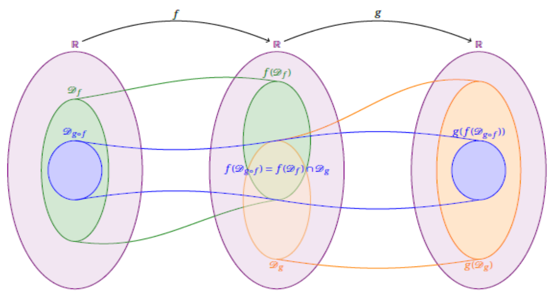
1) If $f\ :\ x \to x^2 + 1$ and $g\ :\ x \to \sqrt{x}$ SO $(g \circ f)(x) := g\big(f(x)\big) = \sqrt{x^2+1}$.

2) The function $u\ :\ t \to \cos(\omega t + \varphi)$ is written as the composite form $u = g \circ f$ with $f\ :\ t \to \omega t + \varphi$ and $g\ :\ x \to \cos x$.

Noticed

In general, we do not necessarily have $f(D_f) \subset D_g$. In this case, the definition set of$g \circ f$ is given by:

$$D_{g \circ f} = \{x \in D_f\ et\ f(x) \in D_g\}.$$



Exercise

Complete the following table (in this exercise we are not interested in the domains of definition).

no.	1	2	3	4	5	6	7
$f(x)$	$x-7$	??	$x+2$	$\frac{x}{x-1}$	??	$\frac{1}{x}$	$\frac{2x+3}{x+7}$
$g(y)$	$\sqrt{y}$	$\sqrt{y-5}$	$3y$	$\frac{y}{y-1}$	$1+\frac{1}{y}$	??	??
$g[f(x)]$	??	$\sqrt{x^2-5}$	??	??	x	x	x

Correction

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[rr, bend left, dashed, "g \circ f"] & \mathbb{R} \arrow[r, "g"] & \mathbb{R} \\ x \arrow[r, mapsto] & y = f(x) \arrow[r, mapsto] & z = g(y) = g(f(x)) \end{tikzcd}$$

Recall

1) $x \xrightarrow{f} y = x - 7 \xrightarrow{g} z = \sqrt{y}$ $\qquad$ by replacing we get $\ g \circ f : x \to g\big(f(x)\big) = \sqrt{x-7}$.

3) $x \xrightarrow{f} y = x + 2 \xrightarrow{g} z = 3y$ by replacing we get
$g \circ f : x \to g\big(f(x)\big) = 3x + 6.$
4) $x \xrightarrow{f} y = \frac{x}{x-1} \xrightarrow{g} z = \frac{y}{y-1}$ by replacing we get
$g \circ f : x \to g\big(f(x)\big) = \frac{\frac{x}{x-1}}{\frac{x}{x-1}-1} = \frac{\frac{x}{x-1}}{\frac{1}{x-1}} = x.$
Noticing that $f = g$ we can write $f\big(f(x)\big) = x$ i.e. $f \circ f = Id$, f is said **idempotent**.

2) $x \xrightarrow{f} y = f(x) \xrightarrow{g} z = \sqrt{y-5} = \sqrt{f(x)-5} = \sqrt{x^2-5}$ by comparing we deduce $f(x) = x^2$.
5) $x \xrightarrow{f} y = f(x) \xrightarrow{g} z = 1 + \dfrac{1}{y} = 1 + \dfrac{1}{f(x)} = x$ by comparing we deduce $f(x) = \dfrac{1}{x-1}$.

6) $x \xrightarrow{f} y = \dfrac{1}{x} \xrightarrow{g} z = g(y) = g(\frac{1}{x}) = x = \dfrac{1}{\frac{1}{x}}$ by comparing we deduce $g(y) = \dfrac{1}{y}$.
7) $x \xrightarrow{f} y = f(x) = \dfrac{2x+3}{x+7} \xrightarrow{g} z = g(y) = g(f(x)) = (g \circ f)(x) = x$ we deduce that $g = f^{-1}$.
We therefore seek x such as $y = \dfrac{2x+3}{x+7}$. Solving this equation, we get $x = \dfrac{3-7y}{y-2}$,
that is $g(y) = \dfrac{3-7y}{y-2}$.

Exercise

Consider the functions from $\mathbb{R}$ towards $\mathbb{R}$ defined by

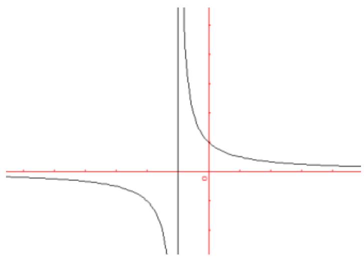
$$u(x) = \frac{1}{1+x} \quad , \quad v(x) = \ln(x) \quad , \quad w(x) = e^x.$$

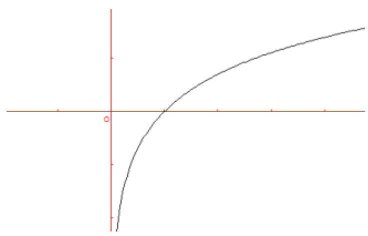
Give the definition set of each of the following functions and write explicitly the expression of the composition:

1) $u \circ v$ **2)** $v \circ u$ **3)** $w \circ v \circ u$

Correction

$u \to u(x) = \dfrac{1}{1+x}$,
Its domain is $D_u = \mathbb{R} \setminus \{-1\}$
and its range is $Im(u) = \mathbb{R}^*$.



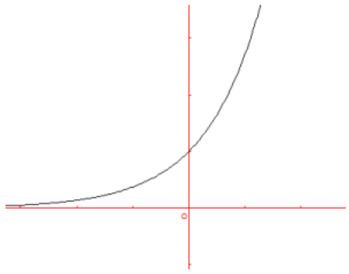


$v \to v(x) = \ln(x)$,
Its domain is $D_v = \mathbb{R}_+^*$
and its range is $Im(v) = \mathbb{R}$.

$w \to w(x) = e^x,$

Its domain is $D_w = \mathbb{R}$

and its range $Im(w) = \mathbb{R}_+^*$.



1)

$$D_{uov} = \{x \in D_v = \mathbb{R}_+^* : v(x) = \ln(x) \in D_u = \mathbb{R} \setminus \{-1\}\}$$
$$= \{x \in \mathbb{R}_+^* : \ln(x) \neq -1\}$$
$$= \mathbb{R}_+^* \setminus \{\frac{1}{e}\}$$
$$u \circ v(x) = u\big(v(x)\big) = u\big(\ln(x)\big) = \frac{1}{1+\ln(x)} \quad , \quad x \in \mathbb{R}_+^* \setminus \{\tfrac{1}{e}\}.$$

2)

$$D_{vou} = \{x \in D_u = \mathbb{R} \setminus \{-1\} : u(x) = \frac{1}{1+x} \in D_v = \mathbb{R}_+^*\}$$
$$= \{x \in \mathbb{R} \setminus \{-1\} : x > -1\}$$
$$=]-1, +\infty[$$
$$v \circ u(x) = v\big(u(x)\big) = v\big(\tfrac{1}{1+x}\big) = \ln(\tfrac{1}{1+x}) \quad , \quad x \in]-1, +\infty[.$$

3)

$$D_{wovou} = \{x \in D_{vou} =]-1, +\infty[: v \circ u(x) = \ln(\frac{1}{1+x}) \in D_w = \mathbb{R}\}$$
$$=]-1, +\infty[$$
$$w \circ v \circ u(x) = w\big(v \circ u(x)\big) = w\big(\ln(\tfrac{1}{1+x}\big) = \tfrac{1}{1+x} \quad , \quad x \in]-1, +\infty[.$$

Question: functions u and $w \circ v \circ u$ are they equal?

(**Answer**: no, they have different domain. $w \circ v \circ u$ is the restriction of u to $]-1, +\infty[$)

Bijection and reciprocal function:

Consider $f : E \to F$.

1) *f is bijective if and only if: it exists $g : F \to E$ such as $f \circ g = Id_F$ and $g \circ f = Id_E$.*

2) *In this case the map g is unique and one-to-one.*

3) *The function g is called* **reciprocal bijection** *of f and is noted f^{-1}.*

we have $\quad (f^{-1})^{-1} = f.$

4) *Whether $f : E \to F$ and $g : F \to G$ are bijective then $g \circ f$ is bijective and we have* $\quad (g \circ f)^{-1} = f^{-1} \circ g^{-1}.$

Example:

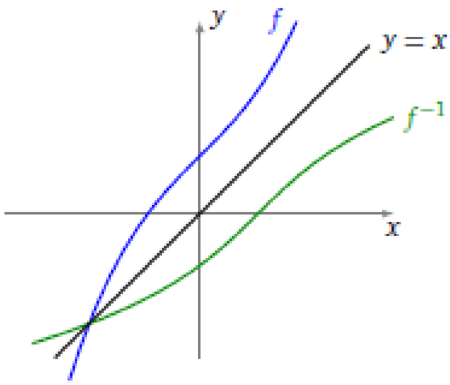
$f:\mathbb{R}\to]0,+\infty[$ defined by $y=f(x)=\exp(x)$ is bijective. Its inverse bijection is $g:]0,+\infty[\to\mathbb{R}$ defined by $x=g(y)=\ln(y)$.

Indeed, we have: $\forall x\in\mathbb{R}\ :\ (g\circ f)(x):=g\big(f(x)\big)=\ln\big(\exp(x)\big)=x$;

and $\forall x\in]0,+\infty[\ :\ (f\circ g)(x):=f\big(g(x)\big)=\exp\big(\ln(x)\big)=x$.

Attention: one may discern between $f^{-1}(x)$ and $[f(x)]^{-1}=\dfrac{1}{f(x)}$.

To remember: For $f:E\subset\mathbb{R}\to F\subset\mathbb{R}$ the curves representative of f And f^{-1} are **symmetrical** with respect to the first bisector



1.4. Variations, parity, periodicity

It is important to **memorize** the general form of curves for the **usual functions** and even more to **know how to read the curve** to easily deduce the properties of these functions.

Curves and direction of variations:

1) *A function f is said to be* **increasing** ***over an interval*** $I\subset D_f$ *if*

$$\forall a,b\in I\ :\ a<b\implies f(a)<f(b).$$

The ***curve*** *of an increasing function is* ***ascending****.*

2) *A function f is said to be* **decreasing** ***over an interval*** $I\subset D_f$ *if*

$$\forall a,b\in I\ :\ a<b\implies f(a)>f(b).$$

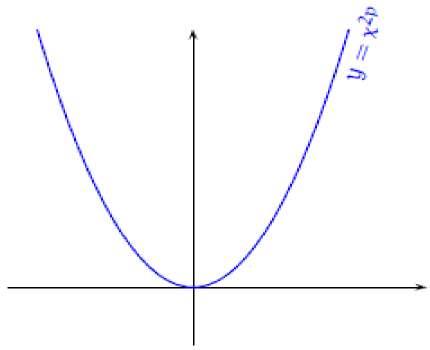
The ***curve*** *of a decreasing function is* ***descending****.*

Example :

If $n = 2p$, $p \in \mathbb{N}^*$, function

$$f : x \in \mathbb{R} \to x^n \text{ , } n \in \mathbb{N}^*$$

is decreasing on$]-\infty, 0)$ and increasing on $(0, +\infty[$.

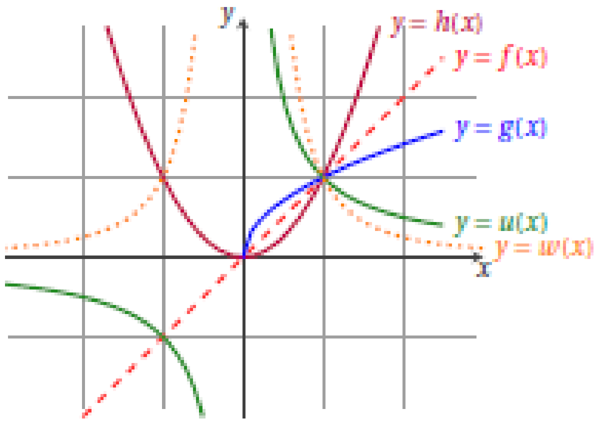


Exercise

Plot on the same graphic the curves of the following functions then give their direction of variations. (Compare the variation with the linear one, in *neighbourhood of* $+\infty$).

$$f : x \to x \quad , \quad g : x \to \sqrt{x} \quad , \quad h : x \to x^2 \quad , \quad u : x \to \frac{1}{x} \quad , \quad w : x \to \frac{1}{x^2}.$$

Correction



$f : x \to f(x) = x$	is strictly increasing on $]-\infty, +\infty[$. *(It is linear, f(x) varies proportionally to x.)*
$g : x \to g(x) = \sqrt{x}$	is strictly increasing on $]-\infty, +\infty[$. *(Its variation in the neighbourhood of $+\infty$ is slower than a linear function).*
$h : x \to h(x) = x^2$	is strictly decreasing on $]-\infty, 0[$ and strictly increasing on $]0, +\infty[$. *(Its variation in the neighbourhood of $+\infty$ is faster than a linear function)*
$u : x \to u(x) = \dfrac{1}{x}$	is strictly decreasing on$]-\infty, 0[$ and on $]0, +\infty[$.
$w : x \to w(x) = \dfrac{1}{x^2}$	is strictly increasing on$]-\infty, 0[$and strictly decreasing on$]0, +\infty[$.

Curves and parity:

1) *A function f is* **pair** *whether:*

$$x \in D_f \iff -x \in D_f \text{And} f(x) = f(-x).$$

The graph of an even function is symmetric with respect to the axis (y'Oy).

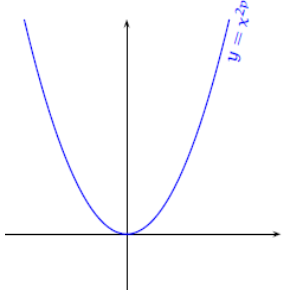
2) *A function f is* **odd** *whether:*

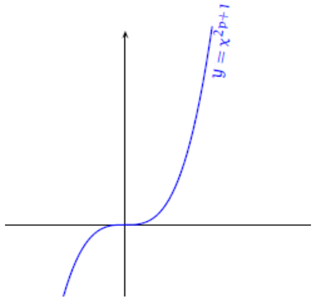
$$x \in D_f \iff -x \in D_f \text{And} f(x) = -f(-x).$$

The graph of an odd function is symmetric with respect to the origin O.

Example:

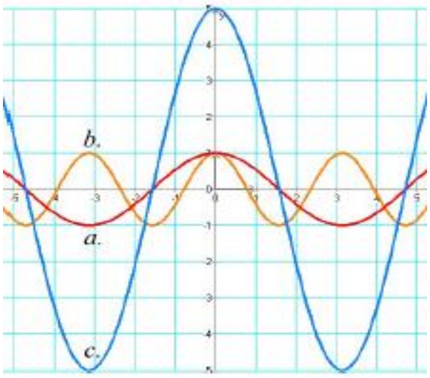
Function $f : x \in \mathbb{R} \to x^n , n \in \mathbb{N}^*$
is even if $n = 2p , p \in \mathbb{N}^*$.
It is odd if $n = 2p + 1 , p \in \mathbb{N}^*$.





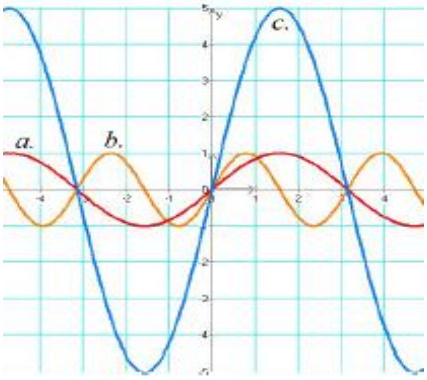
Example:

The cosine function is even



To.: **cos(x) b. : cos(2x) c. : 5cos(x)**

The sine function is odd



To.: **sin(x) b. : sin(2x) c. : 5sin(x)**

Exercise

Study the parity of the following functions?

$$f_1(x) = x^2 - 1 + \sin^2(x) \quad , \quad f_2(x) = \frac{\tan(x) - x}{x^3 \cos(x)} \quad ,$$

$$f_3(x) = \frac{\sin^2(x) - \cos(3x)}{\tan(x)} \quad , \quad f_4(x) = \frac{x-1}{\sin(x+1)} + \cos(x).$$

Correction

1) $f_1(x) = x^2 - 1 + \sin^2(x)$.

$D_{f_1} =]-\infty, +\infty[$, **the domain of** f_1 **is symmetric with respect to the origin i.e.** $x \in D_{f_1} \iff -x \in D_{f_1}$.**and**

$$f_1(-x) = (-x)^2 - 1 + \sin^2(-x) = x^2 - 1 + \sin^2(x) = f_1(x)\,,$$

so, the function f_1 **is even.**

2) $f_2(x) = \dfrac{\tan(x) - x}{x^3 \cos(x)}$. **We have** $x^3 \cos(x) = 0 \iff x = 0 \vee x = (2k+1)\dfrac{\pi}{2}\ ,\ k \in \mathbb{Z}$ **then**

$D_{f_2} = ...\cup] - \frac{5\pi}{2}, -\frac{3\pi}{2}[\cup] - \frac{3\pi}{2}, -\frac{\pi}{2}[\cup] - \frac{\pi}{2}, 0[\cup]0, +\frac{\pi}{2}[\cup] + \frac{\pi}{2}, +\frac{3\pi}{2}[\cup] + \frac{3\pi}{2}, +\frac{5\pi}{2}[\cup...$
the domain of f_2 **is symmetric with respect to the origin i.e.,** $x \in D_{f_1} \iff -x \in D_{f_1}$ **and**

$$f_2(-x) = \frac{\tan(-x) - (-)x}{(-x)^3 \cos(-x)} = \frac{-\tan(x) + x}{-x^3 \cos(x)} = \frac{\tan(x) - x}{x^3 \cos(x)} = f_2(x)\,,$$

We deduce that the function f_2 **is even.**

3) $f_3(x) = \dfrac{\sin^2(x) - \cos(3x)}{\tan(x)}$. **We have** $\tan(x) = 0 \iff x = k\pi\ ,\ k \in \mathbb{Z}$, **then**

$D_{f_3} = ...\cup] - 2\pi, -\pi[\cup] - \pi, 0[\cup]0, +\pi[\cup] + \pi, +2\pi[\cup...$, **the domain of** $E := [-1, 0]$ **is symmetric with respect to the origin**

i.e.,$x \in D_{f_1} \iff -x \in D_{f_1}$ **and**

$$f_3(-x) = \frac{\sin^2(-x) - \cos(-3x)}{\tan(-x)} = \frac{\sin^2(x) - \cos(3x)}{-\tan(x)} = -\frac{\sin^2(x) - \cos(3x)}{\tan(x)} = -f_3(x)\,,$$

the function f_3 **is odd.**

4) $f_4(x) = \dfrac{x-1}{\sin(x+1)} + \cos(x)$. **We have** $\sin(x+1) = 0 \iff x = k\pi - 1\ ,\ k \in \mathbb{Z}$, **the domain of** f_3 **is symmetric with respect to the origin and**

$$f_4(-x) = \frac{-x-1}{\sin(-x+1)} + \cos(-x) = \frac{-x-1}{-\sin(x-1)} + \cos(x) = \frac{x+1}{\sin(x-1)} + \cos(x)\,,$$

the function f_4 **is neither even nor odd.**

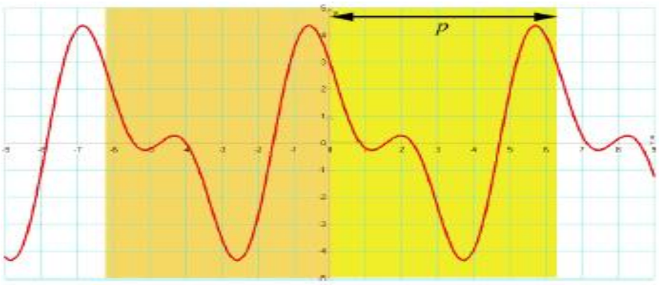
Curves and periodicity:

1) *A function f is* **periodic** ***of period*** $p > 0$ ***(or simply p-periodic) if***

$$\forall x \in D_f\ :\ f(x+p) = f(x).$$

It follows by iteration that: $\forall x \in D_f\ :\ f(x + k\,p) = f(x)\ ,\ k \in \mathbb{Z}$.

The graph of a periodic function repeats every interval of length p units.



Exercise

Calculate the period of the following functions:

$$f_1(x) = \cos(3x) \quad , \quad f_2(x) = \sqrt{\tan(x)} \quad , \quad f_3(x) = \cos^4(8x) \quad , \quad f_4(x) = |\cos(5x)|$$

$$f_5(x) = \cos(3x) + \sin(2x) \quad , \quad f_6(x) = \frac{\cos(5x)}{\sin(5x)} \quad , \quad f_7(x) = \cos(5x)\ \sin(3x) \qquad .$$

Correction

1) $f_1(x) = \cos(3x)$.

$f_1(x+p) = f_1(x) \iff \cos(3x+3p) = \cos(3x) \iff 3p = 2\pi \iff p = \dfrac{2\pi}{3}$.

2) $f_2(x) = \sqrt{\tan(x)}$.

$f_2(x+p) = f_2(x) \iff \sqrt{\tan(x+p)} = \sqrt{\tan(x)} \iff \tan(x+p) = \tan(x) \iff p = \pi$.

3) $f_3(x) = \cos^4(8x)$. **The function** $\cos(8x)$ **is** $\dfrac{\pi}{4}$**- then** $f_3(x) = \cos^4(8x)$ **is also** $\dfrac{\pi}{4}$**-periodic (simple calculation).**

4) $f_4(x) = |\cos(5x)|$.$\cos(5x)$**East**$\dfrac{2\pi}{5}$**-periodic then** $f_4(x) = |\cos(5x)|$ **is also** $\dfrac{\pi}{5}$**-periodic.**

5) $f_5(x) = \cos(3x) + \sin(2x)|$.**We have** $\cos(3x)$ $\dfrac{2\pi}{3}$**-periodic and** $\sin(2x)$ π**-periodic; the period of the sum is**$ppcm(\dfrac{2\pi}{3}, \pi) = 2\pi$

6) $f_6(x) = \frac{\cos(5x)}{\sin(5x)} = \tan(5x)$. **The function** $\tan(x)$ **is** π**-periodic so** $\tan(5x)$ **is** $\dfrac{\pi}{5}$**-periodic.**

7) $f_7(x) = \cos(5x)\ \sin(3x)$. **The function** $\cos(5x)$ **is** $\dfrac{2\pi}{5}$**-periodic and** $\sin(3x)$**is** $\dfrac{2\pi}{3}$**-periodic; the period of the product is** $ppcm(\dfrac{2\pi}{5}, \dfrac{2\pi}{3}) = 2\pi$

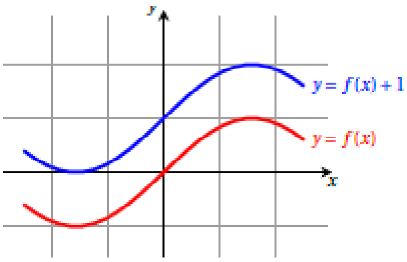
1.5. Translations, expansion, contraction, inversion

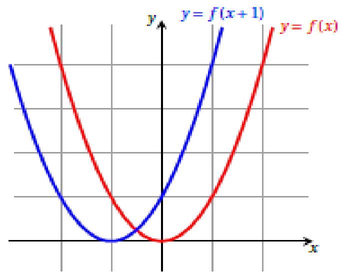
Translations:

Knowing a function f and its curve C_f, we can deduce

***1)**the curve C_g of $x \to g(x) = f(x) + C$, $C \in \mathbb{R}$ by **translating vertically** $C_f$of C units (up for $C > 0$ and down for $C < 0$).*

***2)**the curve C_g of $x \to g(x) = f(x - c)$ $c \in \mathbb{R}$ by **horizontal translation** of C_f of c units (to the right for $c > 0$ and to the left for $c < 0$).*





Expansion, contraction, inversion:

Knowing a function f and its curve C_f, we can deduce

***1)**the curve C_g of $x \to g(x) = K\,f(x)$, $K \in \mathbb{R}$*

***a)** if $K > 1$, by **expanding** vertically by a factor K the curve C_f.*

***b)** if $0 < K < 1$, by **contracting** vertically by a factor K the curve.*

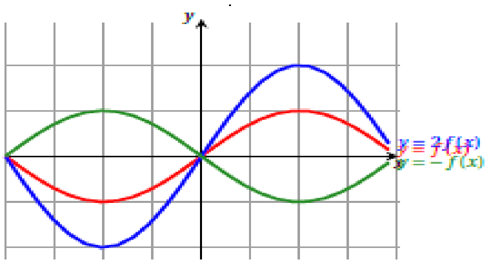
***c)** if $K < 0$, by an **inversion** followed by a **dilation or contraction** of a factor $|K|$ the curve C_f.*

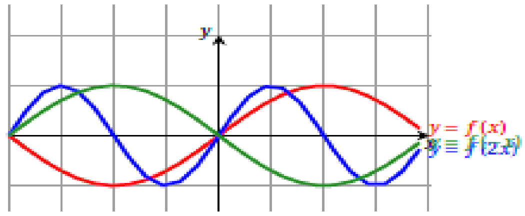
***2)** The curve C_g of $x \to g(x) = f(k\,x)$, $k \in \mathbb{R}$*

***a)** if $k > 1$ by **contracting horizontally** by a factor k the curve C_f.*

***b)** if $0 < k < 1$ by **expanding horizontally** by a factor k the curve C_f.*

***c)** if $K < 0$ by an **inversion** followed by a **dilation or contraction** of a factor $|K|$ the curve C_f.*





Example:

Consider the function f defined on $\mathbb{R}$ by $g(x) = -2x^2 + 4x - 3$.

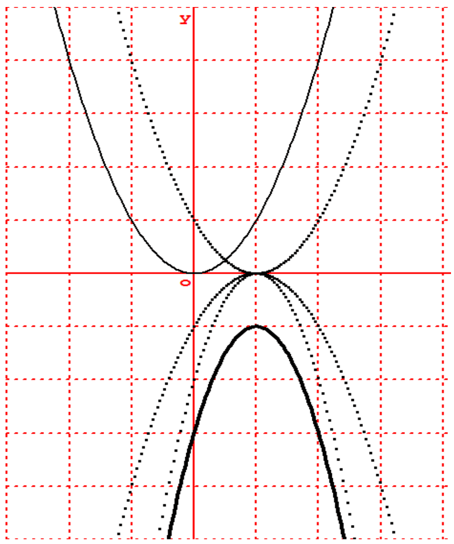
The canonical form of the polynomial of the second degree makes it possible to write

$$g(x) = -2(x-1)^2 - 1.$$

We know the graph of the function f defined by $f(x) = x^2$.

The graph C_g of the function g is obtained from C_f by carrying out the following successive transformations:

1. right translation of 1 unit, $y = (x-1)^2$
2. axial symmetry with respect to (x'ox), $y = -(x-1)^2$
3. vertical expansion by a factor of 2, $y = -2(x-1)^2$
4. translation of 1 unit downwards, $y = -2(x-1)^2 - 1.$



Exercise

For each function, draw the representative curve, then indicate the definition set (domain) and the image set (range):

$f_1(x) = x^2$, $f_2(x) = 1 - x^2$, $f_3(x) = -2(x+1)^2$, $f_4(x) = \sqrt{x}$

$f_5(x) = 3\sqrt{x-1}$, $f_6(x) = \sqrt{3x} - 1$, $f_7(x) = \ln(x)$, $f_8(x) = \ln(-x)$

$f_9(x) = \ln(x-1)$, $f_{10}(x) = \cos(x)$, $f_{11}(x) = \cos(2x)$, $f_{12}(x) = 1 + \cos(2x)$.

$f_{13}(x) = x^2 - 1$, $f_{14}(x) = |x^2 - 1|$, $f_{15}(x) = -1 + |x^2 - 1|$

Correction

$f_1(x) = x^2$

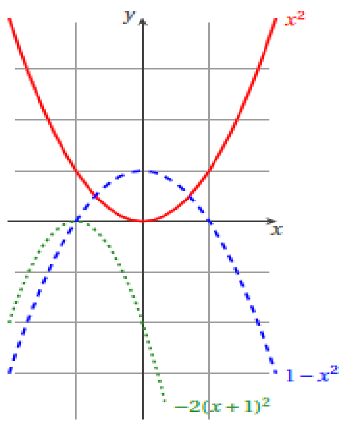
$$D_{f_1} = \mathbb{R} \text{ and } Im(f_1) = [0, +\infty[$$

$f_2(x) = 1 - x^2$

$$D_{f_2} = \mathbb{R} \text{ and } Im(f_2) =]-\infty, +1]$$

$f_3(x) = -2(x+1)^2$

$$D_{f_3} = \mathbb{R} \text{ and } Im(f_3) =]-\infty, +1]$$



$f_4(x) = \sqrt{x}$

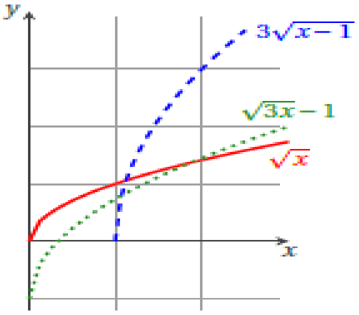
$D_{f_4} = \mathbb{R}_+$ and $Im(f_4) = [0, +\infty[$

$f_5(x) = 3\sqrt{x-1}$

$D_{f_5} = [1, +\infty[$ and $Im(f_5) = [0, +\infty[$

$f_6(x) = \sqrt{3x} - 1$

$D_{f_6} = \mathbb{R}_+$ and $Im(f_4) = [-1, +\infty[$



$f_7(x) = \ln(x)$

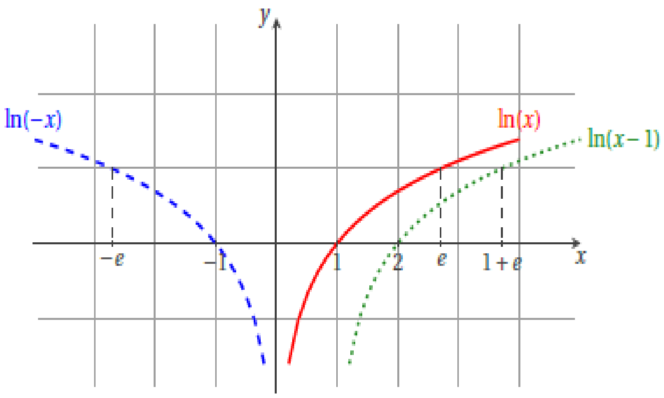
$D_{f_7} = \mathbb{R}_+^*,\ Im(f_1) =]-\infty, +\infty[$

$f_8(x) = \ln(-x)$

$D_{f_8} = \mathbb{R}_-^*,\ Im(f_8) =]-\infty, +\infty[$

$f_9(x) = \ln(x-1)$

$D_{f_9} = [1, +\infty[,\ Im(f_9) =]-\infty, +\infty[$



$f_{10}(x) = \cos(x)$

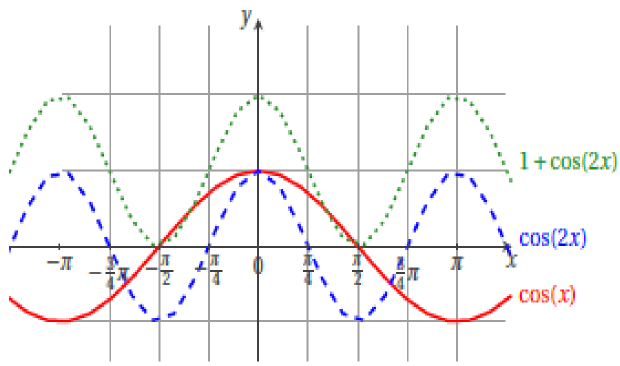
$D_{f_{10}} = \mathbb{R}$, $Im(f_{10}) = [-1, +1]$, $p = 2\pi$.

$f_{11}(x) = \cos(2x)$

$D_{f_{11}} = \mathbb{R}$, $Im(f_{11}) = [-1, +1]$, $p = \pi$.

$f_{12}(x) = 1 + \cos(2x)$

$D_{f_{12}} = \mathbb{R}$, $Im(f_{12}) = [0, +2]$, $p = \pi$.



$f_{13}(x) = x^2 - 1$

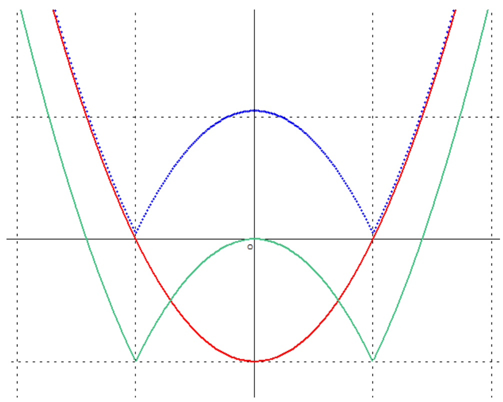
$D_{f_{13}} = \mathbb{R}$ and $Im(f_{13}) = [-1, +\infty[$

$f_{14}(x) = |x^2 - 1|$

$D_{f_{14}} = \mathbb{R}$ and $Im(f_{14}) = [0, +\infty[$

$f_{15}(x) = -1 + |x^2 - 1|$

$D_{f_{15}} = \mathbb{R}$ and $Im(f_{13}) = [-1, +\infty[$



4th chap. : Limits and Continuity

1 Limits of a function

Limit at a point (intuitive definition):

Let $f \; : \; D_f \subset \mathbb{R} \to \mathbb{R}$ be a function.

*1) We say that f admits l for limit at x_0 if the **distance** between $f(x)$ and l is as small as we want l as soon as x is **close** enough to x_0.*

*2) We say that f admits $+\infty$ for limit at x_0 whether $f(x)$ is **greater and greater** as soon as the **distance** between x and x_0 is small enough.*

*3) We say that f admits $-\infty$ for limit at x_0 whether $f(x)$ is **smaller and smaller** as the **distance** between x and x_0 is small enough.*

1.1. Notion of distance

We define a distance in $\mathbb{R}^n$as following:

Definition: (distance)

1) *A **distance** in $\mathbb{R}^n$ is a map $d \; : \; \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty[$ such that*

i) $\forall x, y \in \mathbb{R}^n \qquad d(x,y) = d(y,x)$ *symmetry*

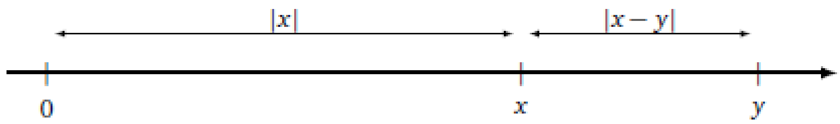
ii) $\forall x, y \in \mathbb{R}^n \qquad d(x,y) = 0 \iff x = y$ *separation*

iii) $\forall x, y, z \in \mathbb{R}^n \qquad d(x,z) \leq d(x,y) + d(y,z)$ *triangular inequality*

2) $d(x,0) := \|x\|$ *is the **norm** of $x \in \mathbb{R}^n$.*

In particular the "***Euclidean" distance***" is defined in $\mathbb{R}^n$:

n=1: $x, y \in \mathbb{R} \qquad d(x,y) := \sqrt{(x-y)^2} = |x-y|$



n=2:

$$x = (x_1, x_2) \; , \; y = (y_1, y_2) \; \in \mathbb{R}^2 \qquad d(x,y) := \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

n=3: $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)$ $\in \mathbb{R}^2$ $\quad d(x,y):=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}$

1.2. Notion of limit by neighbourhoods

We agree to call ***neighbourhoods*** of $a\in\mathbb{R}^n$ sets

$$V(a)=\{x\in\mathbb{R}^n \ : d(x,x_0))<r \ , \ r>0\}.$$

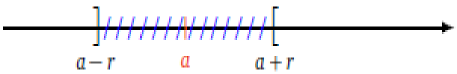
NB: The notion of neighbourhood of a point is very useful in analysis.

In fact, a ***neighbourhood*** of **a** is any ***set containing a ball*** cantered at **a**.

In particular a neighbourhood of $a\in\mathbb{R}^n$ is:

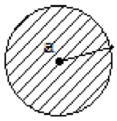
n=1: Any ***interval*** cantered at $a\in\mathbb{R}$ and radius $r>0$:

$x\in V(a) \iff \exists r>0 \ : \ |x-a|<r \iff \exists r>0 \ : \ a-r<x<a+r$



n=2: Any ***disk*** cantered at $a=(a_1,a_2)$ and radius $r>0$

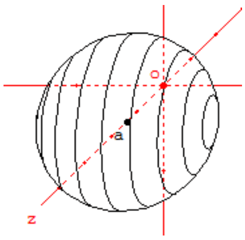
$x\in V(a) \iff \exists r>0 \ : \ d(x,a):=\sqrt{(x_1-a_1)^2+(x_2-a_2)^2}<r$



disc in $\mathbb{R}^2$

n=3: any ***ball*** cantered at $a=(a_1,a_2,a_3)$ and radius $r>0$

$x\in V(a) \iff \exists r>0 \ : \ d(x,a):=\sqrt{(x_1-a_1)^2+(x_2-a_2)^2+(x_3-a_3)^2}<r.$



ball in $\mathbb{R}^3$

Limit at a point (rigorous definition with neighbourhoods):

Let $f \ : \ D_f\subset\mathbb{R}\to\mathbb{R}$ be a function and $l\in\mathbb{R}$.

1) *We say that the limit of f when x tends to x_0 is l*

(*we write* $\lim_{x\to x_0} f(x)=l$ *or* $f(x)\xrightarrow[x\to x_0]{} l$) if:

for every $\varepsilon>0$, there is a neighbourhood V of x_0 ***(interval cantered at x_0)***

such that $\quad x\in V \implies d(f(x),l)<\varepsilon.$

2) *We say that the limit of f when x tends to x_0 Eis $+\infty$*

(*we write* $\lim\limits_{x \to x_0} f(x) = +\infty$ *or* $f(x) \xrightarrow[x \to x_0]{} +\infty$) if:

for every $A > 0$, *there is a neighbourhood* V *of* x_0 **(interval cantered at** x_0)

such that $$x \in V \implies f(x) > A.$$

3) *We say that the limit of f when x tends to x_0 is $-\infty$*

(*we write* $\lim\limits_{x \to x_0} f(x) = -\infty$ *or* $f(x) \xrightarrow[x \to x_0]{} -\infty$) if:

for every $A > 0$, *there is a neighbourhood* V *of* x_0 **(interval cantered at** x_0)

such that $$x \in V \implies f(x) < -A.$$

1.3. Other limits

Limits at infinity:

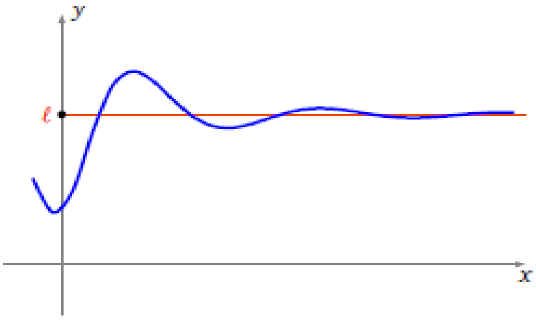
Let $f : D_f \subset \mathbb{R} \to \mathbb{R}$ *be a function and* $l \in \mathbb{R}$.

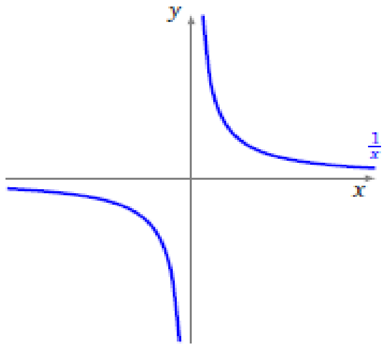
1) $\lim\limits_{x \to +\infty} f(x) = l$ if and only if:

$\forall\, \varepsilon > 0$, *it exists* $A > 0$ *such as* $x > A \implies |f(x) - l| < \varepsilon$.

2) $\lim\limits_{x \to -\infty} f(x) = l$ if and only if:

$\forall\, \varepsilon > 0$, *it exists* $A > 0$ *such as* $x < -A \implies |f(x) - l| < \varepsilon$.



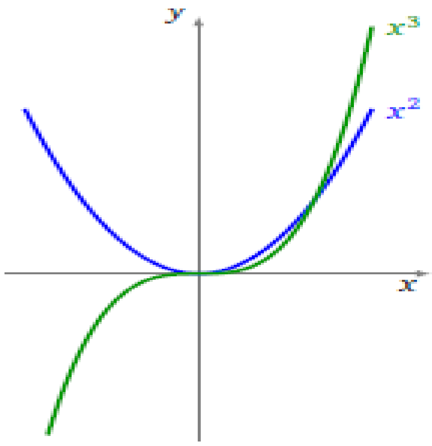


3) $\lim\limits_{x \to +\infty} f(x) = +\infty$ if and only if:

$\forall\, A > 0$, *it exists* $B > 0$ *such a s* $x > A \implies f(x) > B$.

4) $\lim\limits_{x \to +\infty} f(x) = -\infty$ if and only if:

$\forall\, A > 0$, *it exists* $B > 0$ *such as* $x > A \implies f(x) < -B$.



Limits in one direction:

Let $f : D_f \subset \mathbb{R} \to \mathbb{R}$ *be a function; in* $\mathbb{R}$ *there are only two possible directions to go to a point* $x_0 \in \mathbb{R}$*, hence the two definitions:*

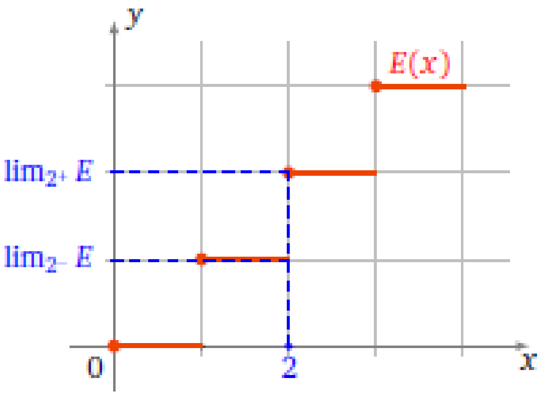
1) $\lim\limits_{x \overset{<}{\longrightarrow} x_0} f(x) = l$ if and only if:

$\forall\ \varepsilon > 0$, *it* exists $\delta > 0$ *such as*

$-\delta < x - x_0 < 0 < \delta \implies |f(x) - l| < \varepsilon$.

2) $\lim\limits_{x \overset{>}{\longrightarrow} x_0} f(x) = l$ if and only if:

$\forall\ \varepsilon > 0$, *it* exists $\delta > 0$ *such as* $0 < x - x_0 < +\delta < \delta \implies |f(x) - l| < \varepsilon$.



Proposition:

Let $f : D_f \subset \mathbb{R} \to \mathbb{R}$ *be a function*

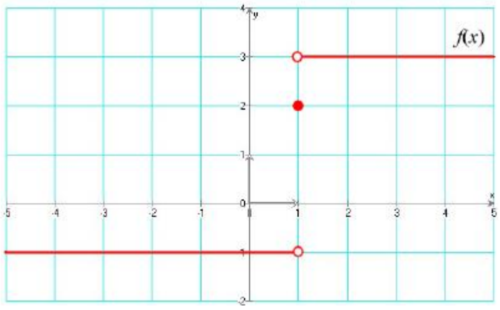
$\lim\limits_{x \to x_0} f(x) = l$ if and only if $\lim\limits_{x \overset{<}{\longrightarrow} x_0} f(x) = l$ *any* $\lim\limits_{x \overset{>}{\longrightarrow} x_0} f(x) = l$.

Exercise

From the drawing give:

1) the expression of $f(x)$

2) the limits at the point $x_0 = 1$ and its image.



Correction

1) $f(x) = \begin{cases} -1 & \text{si } x < 1 \\ 2 & \text{si } x = 1 \\ +3 & \text{si } x > 1 \end{cases}$

2) $\lim\limits_{x \overset{<}{\longrightarrow} 1} f(x) = -1, \lim\limits_{x \overset{>}{\longrightarrow} 1} f(x) = +3$ and $f(1) = +2.$

1.4. Limit properties

Proposition (Unicity of the limit):

If a limit exists then it is unique.

Property (Operations and Limits):

Let f_1 and f_2 be two functions defined close to $x_0 \in \overline{\mathbb{R}}$, such as $f_1(x) \underset{x \to x_0}{\longrightarrow} l_1 \in \overline{\mathbb{R}}$ and $f_2(x) \underset{x \to x_0}{\longrightarrow} l_2 \in \overline{\mathbb{R}}$, then by agreeing in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$ we have

- $f_1(x) + f_2(x) \underset{x \to x_0}{\longrightarrow} l_1 + l_2$ *except for the case* $-\infty + \infty$ *(which is an indeterminate case).*
- $f_1(x) \times f_2(x) \underset{x \to x_0}{\longrightarrow} l_1 \times l_2$ *except for the case* $0 \times \infty$ *(which is an indeterminate case).*
- $\forall \lambda \in \mathbb{R} : (\lambda f_i(x)) \underset{x \to x_0}{\longrightarrow} \lambda l_i$.
- $l_i \neq 0$ *and* $f_i(x) \neq 0$ *close to* x_0 *then* $\frac{1}{f_i(x)} \underset{x \to x_0}{\longrightarrow} \frac{1}{l_i}$.
- $l_2 \neq 0$ *and* $f_2(x) \neq 0$ *close to* x_0 *then* $\frac{f_1(x)}{f_2(x)} \underset{x \to x_0}{\longrightarrow} \frac{l_1}{l_2}$ *except for cases* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ *(undetermined cases).*
- *If* $f(x) \underset{x \to x_0}{\longrightarrow} l \in \overline{\mathbb{R}}$ *and* $g(x) \underset{x \to l}{\longrightarrow} L \in \overline{\mathbb{R}}$, *then* $(g \circ f)(x) = g(f(x)) \underset{n \to x_0}{\longrightarrow} L$

Example:

Let $u : x \to u(x)$ be a function and $x_0 \in \mathbb{R}$ such as $u(x) \xrightarrow[x \to x_0]{} 2$. Pose

$$f(x) = \sqrt{1 + \frac{1}{u^2(x)} + \ln u(x)}\ .$$

If it exists, what is the limit of f in x_0?

Correction

First, let's calculate separately the limits inside the radical:

1) We have $u^2(x) \xrightarrow[x \to x_0]{} 4$*; so close to* x_0 *we have* $u^2(x) \neq 0$ *and then* $\frac{1}{u^2(x)} \xrightarrow[x \to x_0]{} \frac{1}{4}$.

2) Similarly, $u(x) \xrightarrow[x \to x_0]{} 2$ *then, close to* x_0 $u^2(x) > 0$ *then* $\ln u(x)$ *is well defined and* $\ln u(x) \xrightarrow[x \to x_0]{} \ln 2$.

3) $1 + \frac{1}{u^2(x)} + \ln u(x) \xrightarrow[x \to x_0]{} 1 + \frac{1}{4} + \ln 2 = \frac{5}{4} + \ln 2$

4) Finally, the composition rule gives

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \sqrt{1 + \frac{1}{u^2(x)} + \ln u(x)} = \sqrt{\frac{5}{4} + \ln 2}$$

1.5. Calculation of limits

Theorem (Comparisons):

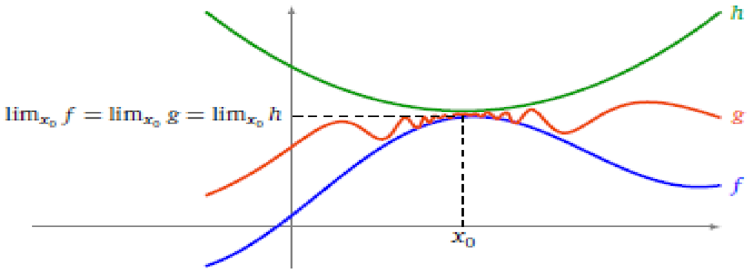
Let f_1 *and* f_2 *be two functions defined close to* $x_0 \in \overline{\mathbb{R}}$*, as* $f_1(x) \xrightarrow[x \to x_0]{} l_1 \in \overline{\mathbb{R}}$ *and* $f_2(x) \xrightarrow[x \to x_0]{} l_2 \in \overline{\mathbb{R}}$ *then*

- *if close to* x_0 *we have* $f_1(x) \leq f_2(x)$ *then* $l_1 \leq l_2$.
- *if a function* g *defined close to* x_0 *check*

$$f_1(x) \leq g(x) \leq f_2(x) \quad \text{and} \quad l_1 = l_2$$

then for $x \to x_0$ *the limit of* $g(x)$ *exists and we have*

$$\lim_{x \to x_0} g(x) = l_1 = l_2.$$



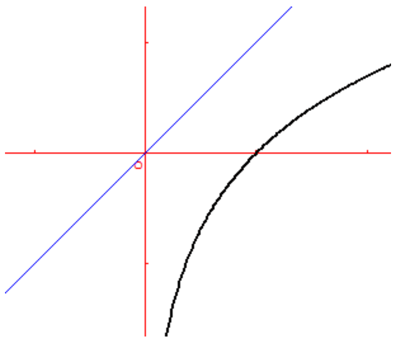
Attention: the ***strict inequalities*** become ***wide inequalities*** after passing to the limit. For example,

$$\forall x \in V_{x_0} : f(x) < g(x) \implies \lim_{n\to x_0} f(x) \leq \lim_{n\to x_0} g(x).$$

Exercise

Demonstrate the following limits: **1)** $\lim_{x\to+\infty} \frac{\ln(x)}{x} = 0$ **2)** $\lim_{x\to+\infty} \frac{\exp(x)}{x} = +\infty$

Correction



1) The curve of the function$x \to \ln(x)$ is located below the 1st bisector; consequently $\forall\, x > 0 \;:\; \ln(x) < x$ so that $\frac{\ln(x)}{x} < 1$. We deduce

$$0 \leq \frac{\ln(x)}{x} \leq \frac{\ln(\sqrt{x}^2)}{\sqrt{x}\,\sqrt{x}} = \frac{\ln(\sqrt{x})}{\sqrt{x}}\frac{2}{\sqrt{x}} < \frac{2}{\sqrt{x}}.$$

The comparison theorem allows us to deduce $0 \leq \lim_{x\to+\infty} \frac{\ln(x)}{x} \leq \lim_{x\to+\infty} \frac{2}{\sqrt{x}} = 0.$

2) We put $u = \exp x$, then $u \xrightarrow[x\to+\infty]{} +\infty$ and we get

$$0 = \lim_{u\to+\infty} \frac{\ln(u)}{u} = \lim_{x\to+\infty} \frac{x}{\exp x} \implies \lim_{x\to+\infty} \frac{\exp x}{x} = +\infty.$$

Definition (Equivalence, domination)

Let f and g be two functions defined on $\mathbb{R}$ and $x_0 \in \overline{\mathbb{R}}$.

1)*We say that f is* **equivalent to** *g close of x_0 if there is a neighbourhood V_{x_0} of x_0 and a function $\varphi : V \to \mathbb{R}$ checking*

$$\forall\, x \in V : \;\; f(x) = g(x)\big(1 + \varphi(x)\big) \qquad \textit{with} \quad \lim_{x\to x_0} \varphi(x) = 0.$$

If $g(x)$ does not vanish in V_{x_0}, we can write more simply:

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=1. \qquad \textit{We note } f \underset{x_0}{\sim} g.$$

2)We say that f is **negligible compared to** *g close of x_0 if there is a neighbourhood V_{x_0} of x_0 and a function $\varphi : V \to \mathbb{R}$ checking*

$$\forall\, x \in V:\ f(x)=g(x)\ \varphi(x) \qquad \textit{with } \lim_{x\to x_0}\varphi(x)=0.$$

If $g(x)$ does not vanish in V_{x_0}, we can write more simply:

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=0. \qquad \textit{We note } f \underset{x_0}{=} \circ(g).$$

Properties (Equivalence, domination)

1) *If $f_1 \underset{x_0}{\sim} \circ(g_1)$ and $f_2 \underset{x_0}{\sim} \circ(g_2)$ (which do not vanish close of x_0) then*

$$f_1\times f_2 \underset{x_0}{\sim} \circ(g_1\times g_2) \qquad \textit{and} \qquad \frac{f_1}{f_2}\underset{x_0}{\sim}\circ\left(\frac{g_1}{g_2}\right).$$

2) $f_1 \underset{x_0}{=} \circ(g)$ *and* $f_2 \underset{x_0}{=} \circ(g)$ *then* $\qquad f_1+f_2 \underset{x_0}{=} \circ(g)$

Attention: *equivalence is not preserved by addition or composition.*

Example:

1) we have $x^2+3x \underset{+\infty}{\sim} x^2$ but $\lim\limits_{x\to+\infty}\frac{e^{x^2+3x}}{e^{x^2}}=\lim\limits_{x\to+\infty}e^{3x}=+\infty\neq 1$ and $e^{x^2+3x}\underset{+\infty}{\nsim} e^{x^2}$

2) $f(x)=x^2+3x \underset{+\infty}{\sim} x^2$ and $g(x)=1-x^2 \underset{+\infty}{\sim} -x^2$ but we clearly see that $f(x)+g(x)=3x-1 \underset{+\infty}{\nsim} (x^2)+(-x^2)=0$.

Examples:(*to remember*)

1) For a ***polynomial function*** $P(x)=a_0+a_1x+a_2x^2+\ldots+a_nx^n$ we have

$$P(x)\underset{\pm\infty}{\sim} a_nx^n \qquad \text{and} \qquad P(x)\underset{0}{\sim} a_0\ (si\ a_0\neq 0)$$

2)For a ***rational function*** $F(x)=\dfrac{a_0+a_1x+a_2x^2+\ldots+a_nx^n}{b_0+b_1x+b_2x^2+\ldots+b_px^p}$ $(a_n,b_p\neq 0)$ we have

$$F(x)\underset{\pm\infty}{\sim}\frac{a_nx^n}{b_px^p} \qquad \text{and} \qquad P(x)\underset{0}{\sim}\frac{a_0}{b_0}\ (si\ a_0,b_0\neq 0)$$

3)*Trigonometric functions*:

$$\sin x \underset{0}{\sim} x \quad , \qquad \tan x \underset{0}{\sim} x \quad , \qquad 1-\cos x \underset{0}{\sim} \frac{x^2}{2}.$$

4)*Logarithm, exponential, power functions*

$$\ln(1+x) \underset{0}{\sim} x, \quad e^x - 1 \underset{0}{\sim} x, \qquad a^x - 1 \underset{0}{\sim} x \ln a, \quad (1+x)^\alpha - 1 \underset{0}{\sim} \alpha\, x.$$

5)For $k>0$, $\alpha, \beta > 0$ (***"logarithm << power << exponential"***)

$$(\ln x)^\beta \underset{+\infty}{=} o(x^\alpha) \;\; i.e. \;\; \lim_{x\to+\infty} \frac{(\ln x)^\beta}{x^\alpha} = 0 \quad \text{and} \quad x^\alpha \underset{+\infty}{=} o(e^{kx}) \;\; i.e. \;\; \lim_{x\to+\infty} \frac{x^\alpha}{e^{kx}} = 0.$$

Notice: Note that $\quad f \underset{x_0}{\sim} g \iff f \underset{x_0}{=} g + \circ(g);$

hence, to find an equivalent to f it has to be written in the form of ***dominant term*** + ***negligible term*** ...

Example: $x^2+3x-1 \underset{+\infty}{=} x^2 + o(x^2) \underset{+\infty}{\sim} x^2$ because $\lim_{x\to+\infty} \frac{3x-1}{x^2} = 0$ **i.e.,** $3x-1 = \circ(x^2)$.

Examples:(to remember)

From the previous remark we can write:

a) *Trigonometric functions:*

$$\sin x \underset{0}{=} x + \circ(x) \quad , \qquad \tan x \underset{0}{=} x + \circ(x) \quad , \qquad \cos x \underset{0}{=} 1 - \frac{x^2}{2} + \circ(x^2).$$

b) *Logarithm, exponential, power functions*

$$\ln(1+x) \underset{0}{=} x + \circ(x), \qquad e^x \underset{0}{=} 1 + x + \circ(x),$$

$$a^x \underset{0}{=} 1 + x \ln a + \circ(x)$$

and

$$(1+x)^\alpha \underset{0}{=} 1 + \alpha\, x + \circ(x).$$

Proposition (Limit calculation by equivalence)

*If f and g are **equivalent close of** x_0 and* $\lim_{n\to x_0} g(x) = l$ *then*

$$\lim_{n\to x_0} f(x) = l.$$

Exercise

Calculate the following limits: **1)** $\lim_{x\to+\infty} \frac{x+2}{x^2+\ln(x)}$ **2)** $\lim_{x\to+\infty} \frac{e^{\sqrt{x}}+2}{x^2+\ln(x)}$

3) $\lim_{x\to+\infty} \frac{e^x - e^{x^2}}{x^2+e^{-x}}$ **4)** $\lim_{x\to+\infty} \frac{x+2}{x^2\ln(x)}$ **5)** $\lim_{x\to0^+} 2x\ln(x+\sqrt{x})$

6) $\lim\limits_{x\to+\infty} \dfrac{x^3-2x^2+2}{x\,\ln(x)}$ **7)** $\lim\limits_{x\to0^+} \dfrac{x^x-1}{\ln(x+1)}$ **8)** $\lim\limits_{x\to-\infty} \dfrac{2}{x+1}\ln\left(\dfrac{x^3+4}{1-x^2}\right)$

9) $\lim\limits_{x\to+\infty} (x^2-1)\ln(7x^3+4x^2+3)$ **10)** $\lim\limits_{x\to2}(-1)^+$ **11)** $\lim\limits_{x\to+\infty} x\ln(x)-x\ln(x+2)$

12) $\lim\limits_{x\to0^+} (1+x)^{\ln(x)}$ **13)** $\lim\limits_{x\to+\infty} \left(\dfrac{x+1}{x-3}\right)^x$ **14)** $\lim\limits_{x\to+\infty} \left(\dfrac{x^3+5}{x^2+2}\right)^{\frac{x+1}{x^2+1}}$

15) $\lim\limits_{x\to+\infty} \left(\dfrac{e^x+1}{x+2}\right)^{\frac{1}{x+1}}$ **16)** $\lim\limits_{x\to0^+} \big(\ln(1+x)\big)^{\frac{1}{\ln(x)}}$ **17)** $\lim\limits_{x\to+\infty} \dfrac{(x+1)^x}{x^{x+1}}$.

Correction

1) $\dfrac{x+2}{x^2+\ln(x)} \underset{+\infty}{\sim} \dfrac{x}{x^2}=\dfrac{1}{x}$ **then** $\lim\limits_{x\to+\infty} \dfrac{x+2}{x^2+\ln(x)}=\lim\limits_{x\to+\infty}\dfrac{1}{x}=0.$

2) $\dfrac{e^{\sqrt{x}}+2}{x^2+\ln(x)} \underset{+\infty}{\sim} \dfrac{e^{\sqrt{x}}}{x^2}$ **then** $\lim\limits_{x\to+\infty} \dfrac{e^{\sqrt{x}}+2}{x^2+\ln(x)}=\lim\limits_{x\to+\infty}\dfrac{e^{\sqrt{x}}}{x^2}=+\infty$ **because**

$(\ln x)^\beta \underset{+\infty}{=} o(x^\alpha)$.

3) $\dfrac{e^x-e^{x^2}}{x^2+e^{-x}} \underset{+\infty}{\sim} \dfrac{-e^{x^2}}{x^2}$ **then** $\lim\limits_{x\to+\infty} \dfrac{e^x-e^{x^2}}{x^2+e^{-x}}=\lim\limits_{x\to+\infty}\dfrac{-e^{x^2}}{x^2}=-\infty$.

4) $\dfrac{x+2}{x^2\,\ln(x)} \underset{+\infty}{\sim} \dfrac{x}{x^2\,\ln(x)}=\dfrac{1}{x\,\ln(x)}$ **then** $\lim\limits_{x\to0^+} \dfrac{x+2}{x^2\,\ln(x)}=\lim\limits_{x\to0^+}\dfrac{1}{x\,\ln(x)}=-\infty.$

5) $2x\ln(x+\sqrt{x})=2x\ln\left(\sqrt{x}(\sqrt{x}+1)\right)=x\ln(x)+2x\ln(\sqrt{x}+1)\xrightarrow[x\to0^+]{} 0$

6) $\dfrac{x^3-2x^2+2}{x\,\ln(x)} \underset{+\infty}{\sim} \dfrac{x^3}{x\,\ln(x)}=\dfrac{x^2}{\ln(x)}$ **then** $\lim\limits_{x\to0^+} \dfrac{x^3-2x^2+2}{x\,\ln(x)}=\lim\limits_{x\to0^+}\dfrac{x^2}{\ln(x)}=+\infty.$

6) We recall that $\ln(1+x)\underset{0}{=}x+\circ(x)$

$$\lim_{x\to0^+}\frac{\ln(3x+1)}{2x}=\lim_{x\to0^+}\frac{3x+\circ(x)}{2x}=\lim_{X\to0^+}\frac{3}{2}+\frac{1}{2}\frac{\circ(x)}{x}=\frac{3}{2}.$$

7) $\dfrac{x^x-1}{\ln(x+1)}=\dfrac{e^{x\ln(x)}-1}{x\ln(x)}\,\dfrac{x}{\ln(x+1)}\,\ln(x)=\dfrac{e^X-1}{X}\,\dfrac{x}{\ln(x+1)}$ **then**

$$\lim_{x\to0^+}\frac{x^x-1}{\ln(x+1)}=\lim_{X\to0}\frac{e^X-1}{X}.\lim_{x\to0^+}\frac{x}{\ln(x+1)}.\lim_{x\to0^+}\ln(x)=-\infty$$

because $\ln(x+1)\underset{0}{\sim}x$ **and** $e^X-1\underset{0}{\sim}X$.

8)

$$\begin{aligned}
\frac{2}{x+1}\ln\Big(\frac{x^3+4}{1-x^2}\Big) &= \frac{2}{x+1}\ln\Big(-x\,\frac{1+4/x^3}{-1/x^2+1}\Big)\\
&= \frac{2}{x+1}\ln(-x) + \frac{2}{x+1}\ln\Big(\frac{1+4/x^3}{-1/x^2+1}\Big)\\
&= \frac{-2x}{x+1}\times\frac{\ln(-x)}{-x} + \frac{2}{x+1}\times\ln\Big(\frac{1+4/x^3}{1-1/x^2}\Big)\\
&\xrightarrow[x\to-\infty]{} -2\times 0 + 0\times 0 = 0\ .
\end{aligned}$$

9)

$$\begin{aligned}
(x^2-1)\ln(7x^3+4x^2+3) &= \frac{x^2-1}{7x^3+4x^2+3}\ (7x^3+4x^2+3)\ln(7x^3+4x^2+3)\\
&= \frac{(x-1)(x+1)}{(x+1)(7x^2-3x+3)}\ X\ln(X)\\
&= \frac{x-1}{7x^2-3x+3}\times X\ln(X)\\
&\xrightarrow[x\to(-1)^+]{X\to 0^+} \frac{-2}{13}\times 0 = 0
\end{aligned}$$

10)

$$\begin{aligned}
(x-2)^2\ln(x^3-8) &= \frac{(x-2)^2}{x^3-8}\ \ (x^3-8)\ln(x^3-8)\\
&= \frac{(x-2)^2}{(x-2)(x^2+2x+4)}\ \ X\ln(X)\\
&= \frac{x-2}{x^2+2x+4}\ \times\ X\ln(X)\\
&\xrightarrow[x\to 2]{X\to 0^+} 0\times 0 = 0
\end{aligned}$$

11)

$$\begin{aligned}
x\ln(x) - x\ln(x+2) &= -x\big(-\ln(x)+\ln(x+2)\big)\\
&= -x\ln\big(\frac{x+2}{x}\big) = -x\ln\big(1+\frac{2}{x}\big)\\
&= -2\ \frac{\ln\big(1+\frac{2}{x}\big)}{\frac{2}{x}} = -2\ \frac{\ln\big(1+X\big)}{X} \xrightarrow[X\to 0]{} -2
\end{aligned}$$

12) Recall that $(1+x)^\alpha \underset{0}{=} 1+\alpha\ x+\circ(x)$**, then**

$$(1+x)^{\ln(x)} = 1 + x\ln(x) + \circ(x) \xrightarrow[x\to 0^+]{} 1.$$

13) Recall that $\ln(1+x) \underset{0}{=} x + \circ(x)$ then

$$\left(\frac{x+1}{x-3}\right)^x = e^{x\ \ln(\frac{x+1}{x-3})} = e^{x\ \ln(\frac{x-3+4}{x-3})} = e^{x\ \ln(1+\frac{4}{x-3})}$$

$$= e^{x\left(\frac{4}{x-3}+\circ(\frac{4}{x-3})\right)} = e^{\frac{4x}{x-3}+\frac{4x}{x-3}\epsilon(x)} \xrightarrow[x\to+\infty]{} e^{4+0} = e^4$$

14)

$$\left(\frac{x^3+5}{x^2+2}\right)^{\frac{x+1}{x^2+1}} = e^{\frac{x+1}{x^2+1}\ \ln(\frac{x^3+5}{x^2+2})} = e^{\frac{x+1}{x^2+1}\ \frac{x^3+5}{x^2+2}\ \frac{\ln(\frac{x^3+5}{x^2+2})}{\frac{x^3+5}{x^2+2}}} = e^{\frac{(x+1)(x^3+5)}{(x^2+1)(x^2+2)}\ \times\ \frac{\ln(X)}{X}}$$

$$\xrightarrow[x\to+\infty]{X\to+\infty} e^{1\times 0} = 1$$

15)

$$\left(\frac{e^x+1}{x+2}\right)^{\frac{1}{x+1}} = e^{\frac{1}{x+1}\ \ln(\frac{e^x+1}{x+2})} = e^{\ \frac{1}{x+1}\ \ln(e^x+1)-\frac{1}{x+1}\ \ln(x+2)}$$

$$= e^{\ \frac{1}{x+1}\ \ln(e^x+1)-\frac{x+2}{x+1}\ \frac{\ln(x+2)}{x+2}} = e^{\ \frac{1}{x+1}\ \ln e^x(1+\frac{1}{e^x})-\frac{x+2}{x+1}\ \frac{\ln(X)}{X}}$$

$$= e^{\ \frac{1}{x+1}\ \ln(e^x)+\frac{1}{x+1}\ \ln(1+\frac{1}{e^x})-\frac{x+2}{x+1}\ \frac{\ln(X)}{X}}$$

$$= e^{\ \frac{x}{x+1}+\frac{\frac{1}{e^x}}{x+1}\ \frac{\ln(1+\frac{1}{e^x})}{\frac{1}{e^x}}-\frac{x+2}{x+1}\ \frac{\ln(X)}{X}}$$

$$= e^{\ \frac{x}{x+1}+\frac{1}{e^x(x+1)}\frac{\ln(1+Y)}{Y}-\frac{x+2}{x+1}\times\frac{\ln(X)}{X}} \xrightarrow[x\to+\infty]{Y\to0,X\to+\infty} e^{1+0\times1+1\times0} = e$$

16) We recall that $\ln(1+x) \underset{0}{=} x + \circ(x)$

$$\Big(\ln(1+x)\Big)^{\frac{1}{\ln(x)}} = e^{\ \frac{1}{\ln(x)}\ \ln\Big(\ln(1+x)\Big)} = e^{\ \frac{\ln(x+\circ(x))}{\ln(x)}} = e^{\ \frac{\ln x(1+\frac{x\epsilon(x)}{x})}{\ln(x)}}$$

$$= e^{1+\frac{\ln(1+\epsilon(x))}{\ln(x)}} \xrightarrow[x\to0^+]{} e$$

17) We recall that $\ln(1+x) \underset{0}{=} x + \circ(x)$

$$\frac{(x+1)^x}{x^{x+1}} = \frac{e^{x\ln(x+1)}}{e^{(x+1)\ln(x)}} = e^{x\ln(x+1)-(x+1)\ln(x)}$$

$$= e^{x\ln x(1+\frac{1}{x})-(x+1)\ln(x)} = e^{x\ln x+x\ln(1+\frac{1}{x})-x\ln(x)-\ln(x)}$$

$$= e^{x\,\frac{1}{x}\,\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}-\ln(x)} = e^{\frac{\ln(1+X)}{X}-\ln(x)} \xrightarrow[x\to+\infty]{X\to0} 0$$

2 CONTINUITY

2.1. Definitions

Definition:

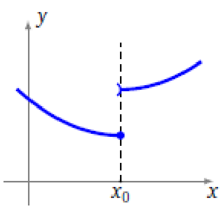
- *We say that a function f is* ***continuous at a point*** $x_0 \in \mathbb{R}$ *if*

$$x_0 \in D_f \quad \textit{and} \quad \lim_{x \to x_0} f(x) = f(x_0).$$

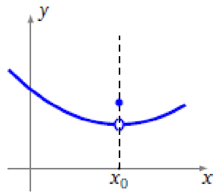
- *We say that f is* ***continues over an interval*** I *if f is continuous at all points* I.

Intuitively: a function is continuous over an interval, if we can draw its graph "without lifting the pencil", **i.e.,** its curve has no jump.

Here are functions that are not continuous in x_0:



CONTINUE LEFT NOT RIGHT



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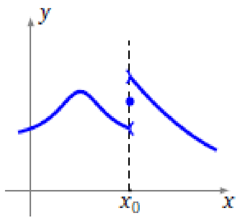


NOR LEFT NOR RIGHT.

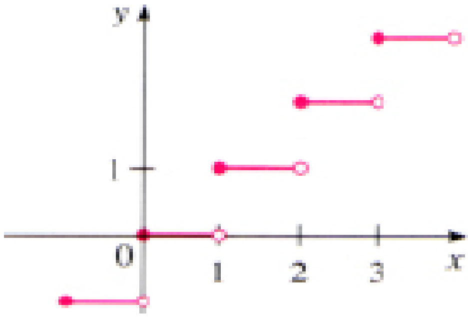
Example:

The ***integer part function***

$x \to f(x) = [x] = E(x) = Ent(x)$

has a discontinuity in each integer value of x because

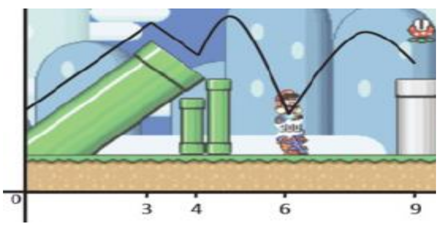
$$\forall n \in \mathbb{Z} \ : \ \lim_{x \overset{<}{\to} n} f(x) \neq \lim_{x \overset{>}{\to} n} f(x)$$



Notice: for every $n \in \mathbb{Z}$, function $x \to f(x) = Ent(x)$ is continuous on the left and discontinuous on the right.

Exercise

In a game Mario runs and jumps to the right. We note x its horizontal position. Its height h is described as a function of x by the following function (defined piecewise):



$$h(x)=\begin{cases}x+3 & \text{si } 0\le x<3\\ 9-x & \text{si } 3\le x<4\\ -4x^2+39x-87 & \text{si } 4\le x<6\\ -x^2+16x-57 & \text{si } 6\le x\le 9\end{cases}$$

Show that Mario's path is continuous on the interval [0; 9]. In other words, check that the function h is continuous.

Correction

For	$0\le x<3$	function $x\to x+3$ is continuous.
For	$3<x<4$	function $x\to 9-x$ is continuous.
For	$4<x<6$	function $x\to -4x^2+39x-87$ is continuous.
For	$6<x\le 9$	function $x\to -x^2+16x-57$ is continuous.

It remains to check if the successive curves meet, **i.e.,** the function h is continuous at $x_1=3$, $x_2=4$ and $x_3=6$.

$\lim\limits_{x\overset{<}{\to}x_1} h(x)=\lim\limits_{x\overset{<}{\to}3} x+3=6$ and $\lim\limits_{x\overset{>}{\to}x_1} h(x)=\lim\limits_{x\overset{>}{\to}3} 9-x=6=h(3)$ so the function h is continuous at $x_1=3$.

$\lim\limits_{x\overset{<}{\to}x_2} h(x)=\lim\limits_{x\overset{<}{\to}4} 9-x=5$ and $\lim\limits_{x\overset{>}{\to}x_2} h(x)=\lim\limits_{x\overset{>}{\to}4} -4x^2+39x-87=5=h(4)$ so the function h is continuous at$x_2=4$.

$\lim\limits_{x\overset{<}{\to}x_3} h(x)=\lim\limits_{x\overset{<}{\to}6} -4x^2+39x-87=3$ and $\lim\limits_{x\overset{>}{\to}x_3} h(x)=\lim\limits_{x\overset{>}{\to}6} -x^2+16x-57=3=h(6)$ so the function h is continuous at $x_3=6$.

Thus, the function h is indeed continuous on the interval [0; 9].

***Exercise** (Discontinuity of the first kind)*

Let f be defined for $x\in\mathbb{R}^*$ by$f(x)=\dfrac{x}{|x|}$. Is-it extendable by continuity at 0?

Correction

One should check if the left and right limits at 0 are equal with the image. We have

$$h(x)=\frac{x}{|x|}=\begin{cases}-1 & \text{si } x<0\\ +1 & \text{si } x>0\end{cases}$$

Clearly $-1=\lim\limits_{x\overset{<}{\to}0} f(x)\neq\lim\limits_{x\overset{>}{\to}0} f(x)=+1$ so the function is discontinuous at 0 and therefore cannot be extended by continuity at 0.

NB: This kind of discontinuity is said to be of the first kind (*limit on the left and limit on the right exist but are different*).

Exercise (Discontinuity of the second kind)

Let f be a map defined by $x \in \mathbb{R}^* \; f(x) = \sin\frac{1}{x}$. Is-it extendable by continuity at 0?

Correction

For $x = \frac{2}{(4n+1)\pi}$ we have $x \to 0 \iff n \to \infty$ and $\lim\limits_{x\to 0} f(x) = \lim\limits_{n\to\infty} \sin\frac{(4n+1)\pi}{2} = +1$

For $x = \frac{2}{(4n+3)\pi}$ we have $x \to 0 \iff n \to \infty$ and $\lim\limits_{x\to 0} f(x) = \lim\limits_{n\to\infty} \sin\frac{(4n+3)\pi}{2} = -1$

For two different paths we had two different limits so the limit does not exist. Therefore, cannot be extended by continuity at 0.

NB: This kind of discontinuity is said to be of the second kind (*limits on the left and/or on the right do not exist*).

Proposition (Examples of continuous functions)

The following functions are continuous over their domains:

- *Polynomial functions (they are continuous in $\mathbb{R}$)*
- *Rational functions (fraction of polynomials)*
- *irrational functions (roots)*
- *trigonometric and hyperbolic functions*
- *reciprocal trigonometric and hyperbolic functions (arcsin, arccos, argsh, argch, arctan ...)*
- *exponential functions*
- *logarithms functions.*

2.2. Continuity and function operations

The elementary operations (addition, multiplication, division by non-zero and composition) preserve continuity.

Proposition

Let f and g be two functions defined on an interval I, and $a \in I$.

If f and g are continuous at a, then

1. *λf is continuous at a ($\forall \lambda \in \mathbb{R}$),* ***2.*** *$f \pm g$ is continuous at a,*

3. *$f \times g$ is continuous at a,* ***4.*** *$\frac{f}{g}$ is continuous if $g(a) \neq 0$.*

5. *If g is continuous at point a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.*

Example: *study the continuity of* $f : x \to f(x) = \frac{\ln(x) + \arctan(x^2+1)}{x^2-1}$.

On $D_f =]0,1[\cup]1,+\infty[$ function $x \to f(x) = \ln(x)$ *is continuous and* function $x \xrightarrow[\text{(continue)}]{\text{polynome}} x^2+1 \xrightarrow[\text{(continue)}]{\arctan(.)} \arctan(x^2+1)$ *is continuous (composed of two continuous functions). So* $x \to \ln(x) + \arctan(x^2+1)$ *is continuous (sum of two continuous functions).*

In the denominator $x \to x^2 - 1$ *is continuous (polynomial).*

We deduce that f is continuous on D_f.

NB: *for a quick answer **we will say more simply that** the function* $f : x \to f(x) = \frac{\ln(x) + \arctan(x^2+1)}{x^2-1}$ *is continuous because **it is composed of continuous functions** (composed in the sense of summation, product, division, composition, etc.).*

2.3. Sequences and Continuity

Proposition (sequential continuity)

Let f be a function defined on an interval I, and $a \in I$.

f is continuous at a if and only if for any sequence $(u_n)_n$ we have

$$u_n \xrightarrow[n\to+\infty]{} a \quad \Longrightarrow \quad f(u_n) \xrightarrow[n\to+\infty]{} f(a).$$

Notice:

This property is intensively used in the study of recurrent sequences $u_{n+1} = f(u_n)$: if f is continuous and $u_n \to l$ then $f(l) = l$.

Consider for example the sequence defined by $u_0 > 0$ and $u_{n+1} = \sqrt{u_n}$.

If the sequence $(u_n)_n$ is convergent then its limit l must verify $l = \sqrt{l}$, that is $l^2 - l = 0$. So, the candidate numbers to be the limit of the sequence $(u_n)_n$ are 0 and 1.

Exercise

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function at 0 such that $\forall x \in \mathbb{R}\ ,\ f(x) = f(2x)$.

Show that f is constant.

Indication: for fixed x study the sequence $f(\frac{1}{2^n}x)$.

Correction

Since $\forall x \in \mathbb{R}\ ,\ f(x) = f(2x)$ we get for fixed $x \in \mathbb{R}$:

$f(\frac{1}{2}x) = f(x)$, $\quad f(\frac{1}{2^2}x) = f(\frac{1}{2}x) = f(x)$, $\; f(\frac{1}{2^3}x) = f(\frac{1}{2^2}x) = f(x)$, ...

$$\forall n \in \mathbb{N} \,,\; f(\tfrac{1}{2^n}x) = f(x).$$

Note $u_n := \frac{1}{2^n}x$, we have $u_n := \frac{1}{2^n}x \xrightarrow[n\to+\infty]{} 0$ and by continuity of f at 0 we get

$$f(0) = \lim_{n\to\infty} f(\tfrac{1}{2^n}x) = f(x).$$

$x \in \mathbb{R}$ being arbitrary, we deduce that $\forall x \in \mathbb{R} \,,\; f(x) = f(0) = cste.$

3 APPLICATIONS OF CONTINUITY

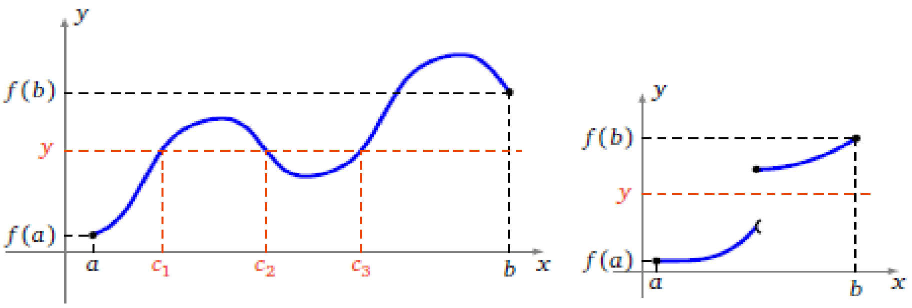
3.1. Theorem: (intermediate values)

Theorem: (intermediate values)

Let f be a continuous function on an interval I of $\mathbb{R}$ and $a, b \in I$ with $f(a) \neq f(b)$, then f reaches all intermediate values between $f(a)$ and $f(b)$.

In other words:

for any value y between $f(a)$ and $f(b)$ there is a value $c \in [a, b]$ such as $f(c) = y$.



INTERMEDIATE VALUE THEOREM (LEFT FIGURE), THE REAL C IS NOT NECESSARILY UNIQUE.
IF THE FUNCTION IS NOT CONTINUOUS, THE THEOREM IS NO LONGER TRUE (RIGHT FIGURE).

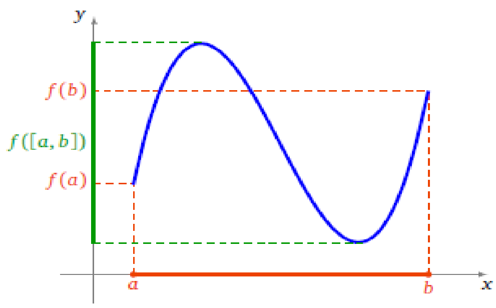
Corollary:

The image of an interval by a continuous function is an interval.

Attention :

It would be wrong to believe that the image of interval [a, b] by a function f is either the interval $[f(a), f(b)]$ or the interval $[f(b), f(a)]$ even when f is continuous (see figure).

For this f will also need to be monotonous



3.1.1. Zeros of a continuous function

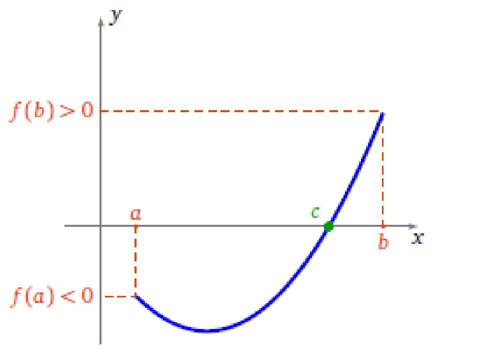
Bolzano's theorem : (zeros of a continuous function)

Consider a ***continuous function*** f *on an subinterval* I *of* $\mathbb{R}$*. If* $a, b \in I$ *verify*

$f(a) \times f(b) < 0$

(i.e. have ***opposite signs****), then there is a value* $c \in [a, b]$ *such that*

$$f(c) = 0.$$



NB: this theorem just guarantees the existence of a zero when the function changes sign.
For ***uniqueness*** we need other assumptions, such as ***monotonicity***.

Exercise:

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^{35} + x - 10^{-35}$. Show that there exists $x \in \mathbb{R}$ such that $f(x) = 0$ and $0 < x < \frac{1}{10}$.

Correction

The function is continuous (*polynomial*), $f(x) = -10^{-35} < 0$ and $f(\frac{1}{10}) = 10^{-35} + \frac{1}{10} - 10^{-35} = \frac{1}{10} > 0$; by the intermediate value theorem there exists (at least one) $x \in]0, 0.1[$ such that $f(x) = 0$.

Exercises

1) Show that the equation $x(1 + e^x) = e^x$ admits a unique solution $l \in (0, 1)$.

2) Consider the function $f : \mathbb{R}_+^* \to \mathbb{R}$ defined by $f(x) = x - 2 + \ln(x)$.

Show that the equation $f(x) = 0$ has a unique solution.

3) Deduce the curve of the following functions from that of $x \to x^2$, then study the number of solutions of the equation $f(x) = 0$ over the interval *I specified* in each case (*without solving the equation*):

a) $f(x) = x^2 - 16$, $I =]0, +\infty[$ b)$f(x) = x^2 - 160$, $I =]-\infty, 0[$

c) $f(x) = x^2 - \sqrt{2}$, $I =]-\infty, +\infty[$.

Fixes

1) *Let be* $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x(1 + e^x) - e^x$ then *x is solution of the equation if and only if* $f(x) = 0$.

This function is continuous $f(0) = -1 < 0$ *and* $f(1) = 1 > 0$, *so by the intermediate value theorem, the function f has* *at least* *one root on* $]0; 1[$ *which solution of the given equation.*

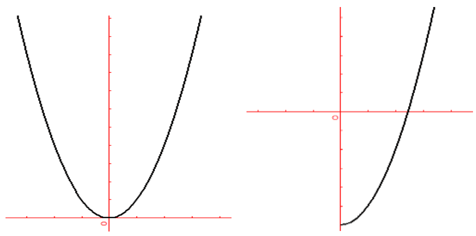
Let us show that f is monotone on $]0;1[$ *to have uniqueness of the solution:*
$f'(x) = 1 + xe^x > 0 \ \forall x \in (0,1)$ therefore the function is strictly increasing on $(0,1)$ and consequently the solution is *unique in this interval.*

2) *The function* $x \to f(x) = x - 2 + \ln(x)$ *is continuous on* $\mathbb{R}_+^* =]0,+\infty[$.
$f'(x) = 1 + \frac{1}{x} > 0 \ \forall x \in]0,+\infty[$ **i.e.,** the function is strictly increasing on $]0,+\infty[$.
On the other hand, $\lim\limits_{\substack{x \to 0 \\ >}} f(x) = -\infty$ *and* $\lim\limits_{x \to +\infty} f(x) = +\infty$ *therefore, according to the intermediate value theorem, the function f admits a root (unique because of monotonicity) in* $]0,+\infty[$.

3)

a) $f(x) = x^2 - 16$, $I =]0,+\infty[$

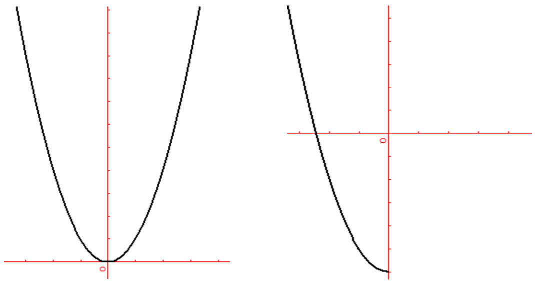
The function f is strictly increasing on $]0,+\infty[$, continuous with $f(0) = -16$ and $\lim\limits_{x \to +\infty} f(x) = +\infty$.



By virtue of the intermediate value theorem, the equation $f(x) = 0$ admits a unique solution on $]0,+\infty[$.

b) $f(x) = x^2 - 160$, $I =]-\infty,0[$

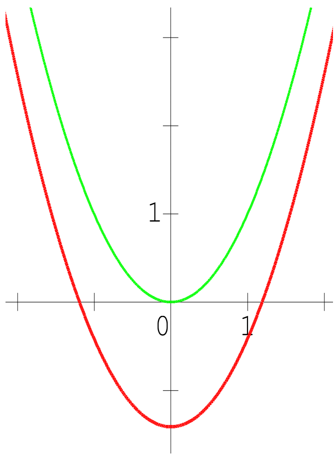
The function f is strictly increasing on $]-\infty,0[$, continuous with $f(0) = -160$ and $\lim\limits_{x \to +\infty} f(x) = +\infty$.



By virtue of the intermediate value theorem, the equation $f(x) = 0$ admits a unique solution on $]-\infty,0[$.

c) $f(x) = x^2 - \sqrt{2}$, $I =]-\infty,+\infty[$

i) The function f is strictly decreasing on $]-\infty,0[$, continuous with $f(0) = -\sqrt{2}$ and $\lim\limits_{x \to -\infty} f(x) = +\infty$.



By virtue of the intermediate value theorem, the equation $f(x) = 0$ admits a unique solution on $]-\infty,0[$.
The function f is strictly increasing on $]0,+\infty[$, continuous with $f(0) = -\sqrt{2}$ and $\lim\limits_{x \to +\infty} f(x) = +\infty$.
By virtue of the intermediate value theorem, the equation $f(x) = 0$ admits a unique solution on $]0,+\infty[$.
In summary, the equation admits two solutions on $I =]-\infty,+\infty[$.

Exercise *(crossing continuous curves)*

Consider f, g two continuous functions from $[a, b]$ into $\mathbb{R}$ such that $f(a) < g(a)$ and $f(b) > g(b)$. Prove that it exists $x_0 \in (a, b)$ such that $f(x_0) = g(x_0)$.

Correction

We consider the function h from $[a, b]$ into $\mathbb{R}$ defined by $h(x) = f(x) - g(x)$.
h is a difference of continuous functions so it is continuous.
We have $h(a) = f(a) - g(a) < 0$ ***and*** $h(b) = f(b) - g(b) > 0$. We deduce from the intermediate value theorem that there exists $x_0 \in (a, b)$ such that $0 = h(x_0) = f(x_0) - g(x_0)$ ***i.e.,*** $f(x_0) = g(x_0)$.

Exercise *(existence of fixed point)*

Let f *be* a continuous function from $[0, 1]$ into $[0, 1]$. Show that it exists $l \in [0, 1]$ such that $f(l) = l$ (*a fixed point* of f).

Correction

If $f(0) = 0$ ***or*** $f(1) = 1$ ***the problem is solved.***
So, suppose $f(0) \neq 0$ so $f(0) > 0$ and $f(1) \neq 1$ ***so*** $f(1) < 1$.
Consider the function $\varphi : [0, 1] \to \mathbb{R}$ such as $\varphi(x) = f(x) - x$. φ is continuous on $[0, 1]$ with $\varphi(0) = f(0) > 0$ and $\varphi(1) = f(1) - 1 < 0$ therefore it exists $l \in]0, 1[$ such that $\varphi(l) = f(l) - l = 0$, **i.e.** $f(l) = l$. This is a fixed point of the function f.

Exercise *(polynomials of odd degree)*

Show that every polynomial of odd degree has at least one real root.

Correction

Let the polynomial of degree ***n*** be odd

$$P(x) = \sum_{k=0}^{n} a_k\, x^k = a_0 + a_1\, x + a_2\, x^2 + \dots + a_n\, x^n$$

The polynomial function $x \to P(x)$ is continuous. We have $a_n \neq 0$ and two cases arise:
1) $a_n > 0$: $\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to +\infty} f(x) = +\infty$ therefore, according to the intermediate value theorem, the ***function admits at least a root in*** $\mathbb{R}$.
2) $a_n < 0$: $\lim\limits_{x \to -\infty} f(x) = +\infty$ and $\lim\limits_{x \to +\infty} f(x) = -\infty$ hence by virtue of the intermediate value theorem, the ***function admits at least a root in*** $\mathbb{R}$.

3.1.2. Dichotomy method (finding zeros of a function)

This is a simple algorithm for finding a zero of a continuous function f over an interval I of $\mathbb{R}$.

We start with two abscissas $a, b \in I$ which surround a zero of the function (***we check by testing the signs of*** $f(a)$ ***and*** $f(b)$***:*** $f(a) \times f(b) < 0$).

At each iteration, we cut the interval into two sub-intervals $[a, c]$ and $[c, b]$, where c is the midpoint of *a* and *b* ($c = (a + b)/2$).

We keep the sub-interval that contains a zero (***depending on the result of the test*** $f(a) \times f(c) < 0$ ***or*** $f(b) \times f(c) < 0$).

Then we cut this sub-interval in two, and so on.

In summary we must proceed as follow :

1. specify the bounds a and $b > a$, the desired precision $\varepsilon > 0$ and give the function f for which a zero is researched.
2. do as long as $b - a > \varepsilon$
   - i) calculate $c = (a + b)/2$,
   - ii) if $f(a) \times f(c) < 0$ then assign to b the value of c ($b \longleftarrow c$),
   - iii) else (**i.e.** $f(b) \times f(c) < 0$) assign to a the value of c ($a \longleftarrow c$).
3. Repeat steps i) , ii) and iii) of the previous phase (as long as $b - a > \varepsilon$).

Example:

For an approximate calculation of $\sqrt{2}$ *nearly* 10^{-5}, *consider the function* $f(x) = x^2 - 2$. *Note that* $\sqrt{2}$ *is a root of the function* f.

Correction

f is continuous on [0,2]. Here is an Excel table that gives after 18 iterations $\sqrt{2} \approx 1.41421$. The calculations of

$$c_k = (a_k + b_k)/2 \ , \ signef(a_k) \ , \ signef(c_k) \ , \ signef(b_k) \ et \ Abs(a_k - b_k)$$

are calculated by Excel via formulas. Just gradually

assign the value of c_k to a_{k+1} if $signef(a_k) = signef(c_k)$

or assign the value of c_k to b_{k+1} if $signef(b_k) = signef(c_k)$.

k	ak	ck	b.k.	sign of f(ak)	sign of f(ck)	sign of f(bk)	Abs(ak-bk)
0	0.00000	1.00000	2.00000	-1	-1	1	2.00000
1	1.00000	1.50000	2.00000	-1	1	1	1.00000
2	1.00000	1.25000	1.50000	-1	-1	1	0.50000
3	1.25000	1.37500	1.50000	-1	-1	1	0.25000
4	1.37500	1.43750	1.50000	-1	1	1	0.12500
5	1.37500	1.40625	1.43750	-1	-1	1	0.06250
6	1.40625	1.42188	1.43750	-1	1	1	0.03125
7	1.40625	1.41406	1.42188	-1	-1	1	0.01563
8	1.41406	1.41797	1.42188	-1	1	1	0.00781
9	1.41406	1.41602	1.41797	-1	1	1	0.00391

10	1.41406	1.41504	1.41602	-1	1	1	0.00195
11	1.41406	1.41455	1.41504	-1	1	1	0.00098
12	1.41406	1.41431	1.41455	-1	1	1	0.00049
13	1.41406	1.41418	1.41431	-1	-1	1	0.00024
14	1.41418	1.41425	1.41431	-1	1	1	0.00012
15	1.41418	1.41422	1.41425	-1	1	1	0.00006
16	1.41418	1.41420	1.41422	-1	-1	1	0.00003
17	1.41420	1.41421	1.41422	-1	-1	1	0.00002
18	1.41421	**1.41421**	1.41422	-1	-1	1	**0.00001**

Exercise

Show that there is $x \in]0,1[$, unique, such that $\arctan(x) = \pi/8$.

Find the value of *x* by dichotomy with an accuracy of 1/10.

Correction

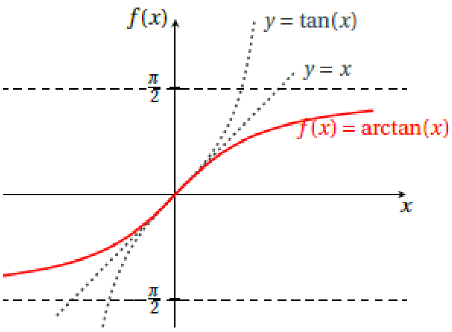
The function $x \to \arctan(x)$ is continuous and strictly increasing from $\mathbb{R}$ onto $]-\pi/2, +\pi/2[$.

We have $\pi/8 \in]-\pi/2, +\pi/2[$ then by mean of the intermediate value theorem, there exist $c \in \mathbb{R}$ (*unique par monotonicity*) such as $\arctan(c) = \pi/8$.

$a_1 = 0 \implies atctan(a_1) = -0,392699082 < 0,$

$b_1 = 1 \implies atctan(b_1) = +0,392699082 > 0.$

Take $f(x) = atan(x) - pi/8$ and $\varepsilon = 10^{-2}$.



	f(xk) < 0		**f(xk) > 0**		
k	***ak***	***xk = (ak + bk)/2***	***bk***	**Error *k* = abs(bk - ak)**	***atan(xk)***
1	0,000	0,500	1,000	1,000	0,0709485
2	0,000	0,250	0,500	0,500	-0,1477204
3	0,250	0,375	0,500	0,250	-0,0339284
4	0,375	0,438	0,500	0,125	0,0197114
5	0,375	0,406	0,438	0,063	-0,0068164
6	0,406	0,422	0,438	0,031	0,0065217
7	0,406	0,414	0,422	0,016	-0,0001289
8	0,414	**0,418**	0,422	**0,008**	0,0032010

3.2. Monotonicity and injectivity for a continuous function

Bijection theorem

If a continuous function f is strictly monotone on an interval I of $\mathbb{R}$, then f is a bijective from I onto $f(I)$.

Moreover, its inverse bijection is continuous and monotone from $f(I)$ onto I and of the same direction of variation as f.

Example

Let $n > 1$ and $f : [0, +\infty[\to [0, +\infty[$ be the function defined by $f(x) = x^n$. f is continuous and strictly increasing. Moreover $f(0) = 0$ and $f(x) \underset{\to +\infty}{\longrightarrow} +\infty$ therefore $Imf = [0, +\infty[$ and $f : [0, +\infty[\to [0, +\infty[$ is one to one (bijective).

Its inverse bijection f^{-1} is denoted: $f^{-1}(x) = x^{\frac{1}{n}}$ or also $f^{-1}(x) = \sqrt[n]{x}$: it is the ***n-th root*** function. It is continuous and strictly increasing.

Exercise

Let be the function $f : \mathbb{R}_+^* \to \mathbb{R}$ defined by $f(x) = x - 2 + \ln(x)$.

1) Show the existence of g the inverse bijection of f.

2) Study the monotony and continuity of g and specify its behavior (limits) at the bounds of the definition set (domain).

Correction

1) *The function* $x \to f(x) = x - 2 + \ln(x)$ *is continuous* $\mathbb{R}_+^* =]0, +\infty[$.

$f'(x) = 1 + \frac{1}{x} > 0 \ \forall x \in]0, +\infty[$ **i.e.,** the function is strictly increasing on $]0, +\infty[$.

Its image set is $]\lim_{x \overset{>}{\longrightarrow} 0} f(x), \lim_{x \longrightarrow +\infty} f(x)[=] - \infty, +\infty[$.

Therefore, *the function* f is one-to-one from $\mathbb{R}_+^*$ to $] - \infty, +\infty[$.

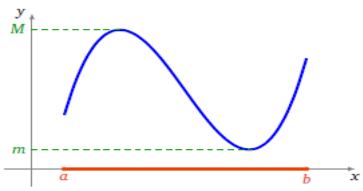
2) f being continuous and monotone (increasing), according to the bijection theorem its converse g is also continuous and monotone (increasing).

On the other hand, $\lim\limits_{x \overset{>}{\longrightarrow} 0} f(x) = -\infty$ involves $\lim\limits_{x \longrightarrow -\infty} g(x) = 0$ *and* $\lim\limits_{x \longrightarrow +\infty} f(x) = +\infty$ implies $\lim\limits_{x \longrightarrow +\infty} g(x) = +\infty$.

3.3. Extreme values theorem

Weierstrass Theorem (of extreme Values) :

A ***continuous*** *function defined on a* ***bounded and closed interval*** *admits a* ***maximum*** *and a* ***minimum*** *on (called "****extreme values****").*



Exercise

Let be $f : \mathbb{R}_+ \to \mathbb{R}$ continuous admitting a finite limit at $+\infty$. Show that f is bounded. Does it reach its limits?

Hint: thanks to the definition of the limit in $+\infty$, we can have a bound on an interval $[A, +\infty[$; then work on $[0, A]$.

Correction

Denote l the finite limit in $+\infty$ and recall that $\lim_{x\to+\infty} f(x) = l$ *if and only if*

$$\forall\ \varepsilon > 0, \text{ it exists } A > 0 \text{ such as } x > A \implies |f(x) - l| < \varepsilon.$$

In particular for $\varepsilon = 1$, there exists $A > 0$ such that:

if $x \in [A, +\infty[$ then $l - 1 < f(x) < l + 1$

which shows that f is bounded in $[A, +\infty[$.

It remains to check boundedness on $[0, A]$. f is continuous on the bounded closed interval $[0, A]$, from the extreme value theorem we deduce that f is bounded, so there exists $m, M \in \mathbb{R}$ such that $\forall\, x \in [0, A]$ we have: $m \le f(x) \le M$.

Therefore, $\forall\, x \in \mathbb{R}_+$

$$\min(l - 1, m) \le f(x) \le \max(l + 1, M),$$

i.e., f is bounded in $\mathbb{R}_+$.

Example: Consider $f : \mathbb{R}_+ \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x+1}$. f is continuous and has a finite limit $l = 0$ at $+\infty$;

we deduce from the above that f is bounded in $\mathbb{R}_+$. f is strictly decreasing in $\mathbb{R}_+$, consequently $\sup_{x\ge 0} f(x) = \max_{x\ge 0} f(x) = f(0) = 1$ (***reached***) and

$\inf_{x\ge 0} f(x) = \lim_{x\to+\infty} f(x) = 0$ (***not reached***).

5th chap. : Derivations - Approximations

1 Derivation

1.1. Definitions

Definition: (Derivation)

- *Let be $I \subset \mathbb{R}$ a non-empty open set. We say that a function $f : I \to \mathbb{R}$ is **differentiable at a point** x_0 (or admits a derivative at x_0) if the rate-increase $\frac{\Delta f(x)}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}$ admits a limit $\Delta x \to 0$, noted $f'(x_0)$, when $\Delta x \to 0$:*

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

- *$f : I \to \mathbb{R}$ is **differentiable on I** if it is differentiable at any point of I.*
- *The function $x \in I \to f'(x)$ is called **derived function** of f and is denoted f' or (in Leibniz notation) $\frac{df}{dx}$.*

Theorem: (differentiability implies continuity)

*Let f be a function defined on an **open interval** $I \subset \mathbb{R}$ and $x_0 \in I$.*

*If f is **differentiable** at x_0 then it is **continuous** at x_0.*

*If f is **differentiable** on I then it is **continuous** on I.*

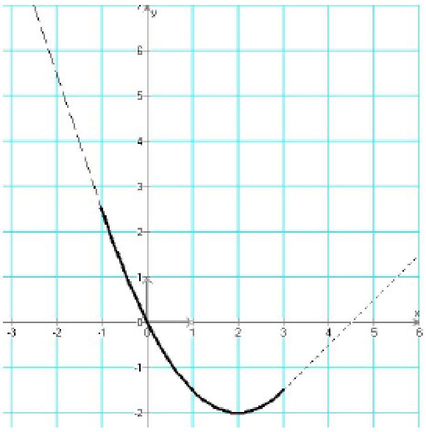
Higher order derivatives

- *For $n \in \mathbb{N}$ we define by induction the **n-th derivative** (or **derivative of order n**) of f by setting $f^{(0)} = f$ then $f^{(n)} = (f^{(n-1)})'$.*
- *We say that f is **of class C^n on** I, and we write $f \in C^n(I)$, when f is n times differentiable on$I$ and the derivative $f^{(n)}$ is continuous on I.*
- *We say that f is **of class C^∞ on** I, and we write $f \in C^\infty(I)$, if f is of class C^n on I, for every $n \in \mathbb{N}$.*

Exercise

We want to extend a parabolic segment by two lines, so that the function obtained is everywhere derivable (*see the opposite drawing*).

Complete the formula below with equations of lines:



$$f(x)=\begin{cases} \dots\dots\dots & \text{for } x<-1 \\ \dfrac{x^2}{2}-2x & \text{for } -1\le x\le 3 \\ \dots\dots\dots & \text{for } x>3 \end{cases}$$

Correction

We must look for real numbers a and b such that:

$$f(x)=\begin{cases} a(x+1)+2,5 & \text{for } x<-1 \\ \dfrac{x^2}{2}-2x & \text{for } -1\le x\le 3 \\ b(x-3)-1,5 & \text{for } x>3 \end{cases}$$

Note that $x\to a(x+1)+2.5$ is differentiable on $]-\infty,1[$; $\quad x\to\dfrac{x^2}{2}-2x$ is differentiable on $]-1,+3[$

and $x\to b(x-3)-1.5$ is differentiable on $]+3,+\infty[$.

It remains that f must be differentiable at points -1 and $+3$.

1) For $x_0=-1$:

Left derivative: $$\lim_{x\xrightarrow{x<-1}-1}\frac{f(x)-2.5}{x+1}=\lim_{x\xrightarrow{x<-1}-1}\frac{a(x+1)}{x+1}=a$$

Right derivative:

$$\lim_{x\xrightarrow{x>-1}-1}\frac{f(x)-2.5}{x+1}=\lim_{x\to-1}\frac{x^2/2-2x-5/2}{x+1}=\lim_{x\to-1}\frac{1}{2}\frac{x^2-4x-5}{x+1}=\lim_{x\to-1}\frac{1}{2}\frac{(x-5)(x+1)}{x+1}=\lim_{x\to-1}\frac{1}{2}(x-5)=-3$$

So, we *must have* $\qquad a=-3.$

2) For $x_0=+3$:

Left derivative:

$$\lim_{x\xrightarrow{<}+3}\frac{f(x)+1.5}{x-3}=\lim_{x\to-1}\frac{x^2/2-2x+3/2}{x-3}=\lim_{x\to+3}\frac{1}{2}\frac{x^2-4x+3}{x-3}=\lim_{x\to-1}\frac{1}{2}\frac{(x-1)(x-3)}{x-3}=\lim_{x\to+3}\frac{1}{2}(x-1)=+1$$

Right derivative: $$\lim_{x\xrightarrow{>}+3}\frac{f(x)+1.5}{x+1}=\lim_{x\to+3}\frac{b(x-3)}{x-3}=b$$

Therefore, we *must* have $\qquad b=+1.$

We deduce $$f(x)=\begin{cases} -3(x+1)+2,5 & \text{for } x<-1 \\ \dfrac{x^2}{2}-2x & \text{for } -1\le x\le 3 \\ +1(x-3)-1,5 & \text{for } x>3 \end{cases}=\begin{cases} -3x-0,5 & \text{for } x<-1 \\ \dfrac{x^2}{2}-2x & \text{for } -1\le x\le 3 \\ x-4,5 & \text{for } x>3 \end{cases}$$

Exercise

a. If a cube with sides of 2 cm increase by 1 cm/min, how does the volume increase?

b. If the area of a sphere with a radius of 10 cm increases by 5 cm2/min, how does the radius increase?

Correction

a) The volume of the cube with side x is $v = x^3$. We have $\frac{\Delta v}{\Delta t} \approx \frac{dv}{dt}$ (recall that $\frac{dv}{dt} := \lim_{t} \frac{\Delta v}{\Delta t}$) hence

$$\frac{\Delta v}{\Delta t} \approx \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} \approx 3x^2 \cdot \frac{\Delta x}{\Delta t} = 3(2)^2 \cdot 1 = 12\ cm^3/min.$$

b) The area of a sphere with a radius of 10 cm r is $s = 4\pi r^2$.

$\frac{\Delta s}{\Delta t} \approx \frac{ds}{dt} = \frac{ds}{dr} \cdot \frac{dr}{dt} \approx 8\pi r \cdot \frac{\Delta r}{\Delta t} = 5,$ we deduce $\cdot \frac{\Delta r}{\Delta t} = \frac{5}{8\pi r} = \frac{5}{8\pi\, 10} = \frac{1}{16\pi}\ cm/min.$

Exercise

A breach opened in the sides of a tanker. Suppose that the petrol extends around the breach according to a disc with a 2 m/s increasing radius. How fast does the surface of the oil slick-disc increase when the radius is 60 m?

Correction

Let A be the area of the disc (in m2), *r* the radius of the disc (in m) and *t* the time (in seconds) elapsed since the accident.

We want to calculate the rate of increase of the polluted area with respect to time, $\frac{\Delta A}{\Delta t} \approx \frac{dA}{dt}$ (remember $\frac{dA}{dt} := \lim_{t} \frac{\Delta A}{\Delta t}$).

We will use the relationship: $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt};$

The rate of increase of the radius is (given) $\frac{dr}{dt} = 2\ m/s.$

Consider the formula $A = \pi r^2$: Deriving with respect to *r*, we get: $\frac{dA}{dr} = 2\pi r$

So that, for $r = 60$ we'll get $\frac{dA}{dr} = 120\pi.$

We deduce the variation of the speed of the surface of the oil spill when the radius of the slick is 60 m

$$\frac{\Delta A}{\Delta t} \approx \frac{dA}{dt} = 120\pi \cdot 2 \approx 754\ m^2/s.$$

1.2. Derivatives of usual functions

$(x^n)' = nx^{n-1}$

$(e^x)' = e^x$

$(a^x)' = a^x \ln(a)$

$(\ln(x))' = \frac{1}{x}$

$(\sin(x))' = \cos(x)$

$(\cos(x))' = -\sin(x)$

$(\tan(x))' = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$

$(\tan(x))' = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$

$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$

$(\arccos(x))' = -\frac{1}{\sqrt{1-x^2}}$

$(\arctan(x))' = \frac{1}{1+x^2}$

$(\sinh(x))' = \cosh(x)$

$(\cosh(x))' = \sinh(x)$

Examples

1) $\left(\frac{1}{x}\right)' = \left(x^{-1}\right)' = -1\, x^{-1-1} = -x^{-2} = \frac{-1}{x^2}$.

2) $\left(\sqrt{x}\right)' = \left(x^{1/2}\right)' = \frac{1}{2}\, x^{1/2-1} = \frac{1}{2}\, x^{-1/2} = \frac{-1}{2\sqrt{x}}$.

3) $\alpha \in \mathbb{R}\ :\ \left(x^\alpha\right)' = \left(e^{\alpha \ln x}\right)' = e^{\alpha \ln x} \times \alpha \frac{1}{x} = \alpha \frac{1}{x} x^\alpha = \alpha\, x^{\alpha-1}$.

4) $\left(2^x\right)' = \left(e^{x \ln 2}\right)' = e^{x \ln 2} \times \ln 2 = \ln 2 \,.\, 2^x$.

1.3. Calculation rules for derivatives

Derivable functions

- ***Elementary functions*** *such as polynomials, rational and irrational functions, exponential, logarithmic, trigonometric and hyperbolic functions* ***are differentiable in their respective domains.***

Derivative of compound functions

- *If f and u are differentiable then the* ***composite function*** *$x \to (f \circ u)(x) = f\big(u(x)\big)$ is* ***differentiable*** *on its domain and we have*

$$(f \circ u)'(x) = \left[f\big(u(x)\big)\right]' = u'(x) \times f'\big(u(x)\big).$$

or (in Leibniz notation easier to remember)

$$\frac{d(f \circ u)(x)}{dx} = \frac{d[f\big(u(x)\big)]}{dx} = \frac{d\,f\big(u\big)}{du}.\frac{du}{dx}.$$

Examples: on domain of U, we have

1) $D_U = \{x \in \mathbb{R} \ : \ U(x) \neq 0\}$: $\left(\frac{1}{U(x)}\right)' = (U^{-1}(x))' = -1\, U'(x)\, U^{-2}(x) = \frac{-U'(x)}{U^2(x)}$.

2) $D_U = \{x \in \mathbb{R} \ : \ U(x) \geq 0\}$: $\left(\sqrt{U(x)}\right)' = \left(U^{1/2}(x)\right)' = \frac{1}{2} U'(x)\, U^{\frac{1}{2}-1}(x) = \frac{U'(x)}{2\sqrt{U(x)}}$.

3) $D_U = \{x \in \mathbb{R} \ : \ U(x) > 0\}$: $\alpha \in \mathbb{R}$, $(U^\alpha(x))' = \alpha\, U'(x)\, U^{\alpha-1}(x)$.

4) $D_U = \mathbb{R}$: $(\sin U(x))' = U'(x) \cos U(x)$.

Examples (derivatives of common composite functions)

$([f(x)]^n)' = n[f(x)]^{n-1} f'(x)$

$(e^{f(x)})' = e^{f(x)} f'(x)$

$(a^{f(x)})' = a^{f(x)} \ln(a) f'(x)$

$(\ln(f(x)))' = \frac{f'(x)}{f(x)}$

$(\sin(f(x)))' = f'(x) \cos(f(x))$

$(\cos(f(x)))' = -f'(x) \sin(f(x))$

$(\tan(f(x)))' = \frac{f'(x)}{\cos^2(f(x))} = 1 + \tan^2(f(x))$

$(\arcsin(f(x)))' = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$

$(\arccos(f(x)))' = -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$

$(\arctan(f(x)))' = \frac{f'(x)}{1+(f(x))^2}$

$(\sinh(f(x)))' = f'(x) \cosh(f(x))$

$(\cosh(f(x)))' = f'(x) \sinh(f(x))$

Rules for calculating the derivative

- *The **sum**, **product** and **quotient**, of differentiable functions is a differentiable function over their domains of definition; and we have for differentiable functions f, g and $\lambda \in \mathbb{R}$:*

$$(f+g)' = f' + g' \ , \qquad (f \times g)' = f' \times g \ + \ f \times g' \ ,$$

$$(\lambda \times f)' = \lambda \times f' \ , \qquad \left(\frac{f}{g}\right)' = \frac{f' \times g \ - \ f \times g'}{g^2} \quad (g(x) \neq 0).$$

- If f and g *are* n-times differentiable then the product $(f.g)$ is ***n-times differentiable*** and we have (***Leibniz formula***)

$$(f \cdot g)^{(n)} = f^{(n)} \cdot g + \binom{n}{1} f^{(n-1)} \cdot g^{(1)} + \cdots + \binom{n}{k} f^{(n-k)} \cdot g^{(k)} + \cdots + f \cdot g^{(n)} \ ,$$

which can be written $$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} \cdot g^{(k)}.$$

Examples

For n = 1 we'll get $(f\ .\ g)' = f'\ .\ g\ +\ f\ .\ g'$,

for n = 2, we'll get $(f\ .\ g)'' = f''\ .\ g\ + 2\ f'\ .\ g' + f\ .\ g''$.

Examples

Compute the n-th derivatives of $\exp(x)\,.\,(x^2+1)$ for all n > 0.

Putting $f(x) = \exp(x)$ we get $f'(x) = \exp(x)$, $f''(x) = \exp(x)$, ...

Denote $g(x) = x^2 + 1$ then $g'(x) = 2x$, $g''(x) = 2$ and for k > 3, $g^{(k)}(x) = 0$.

Applying Leibniz's formula, we'll have

$$\begin{aligned}\exp(x)\,.\,(x^2+1) = (f.g)^{(n)} &= f^{(n)}.g + f^{(n-1)}.g^{(1)} + f^{(n-2)}.g^{(2)} + f^{(n-3)}.0 + f^{(n-4)}.0 + \ldots \sin^2 x / \cos^2 x \\ &= \exp x.(x^2+1) + \exp x.(2x) + \exp x.(2) \\ &= (x^2+1)e^x + 2xe^x + 2e^x = (x^2+2x+3)e^x\end{aligned}$$

Derivative of the reciprocal bijection

- *If a bijection $f : E \to F$ is differentiable then its inverse bijection $f^{-1} : F \to E$ (defined by $y = f^{-1}(x) \iff x = f(y)$) is differentiable and we have*

$$\left(f^{-1}(x)\right)' = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{f'(y)} = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Notice.:

It is easier to find the formula by differentiating $f\left(g(x)\right) = x$ *with* $g = f^{-1}$ *:*

$$[f(g(x))]' = [x]' \iff g'(x) \times f'\left(g(x)\right) = 1 \iff \left(f^{-1}(x)\right)' = g'(x) = \frac{1}{f'\left(g(x)\right)} = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

2 FIRST-ORDER APPROXIMATION

2.1. Linearization - Differentiability

Definition: (differentiability)

If a function f defined on an ***open interval*** $I \subset \mathbb{R}$ admits in a neighbourhood of *a point* $x_0 \in I$ an ***approximation of order 1*** *(or* **linear***)* ***i.e.,*** *that there exists a* ***linear map*** $x \in V_{x_0} \to L(x)$ *such as*

$$f(x) = L(x) + \circ(x - x_0);$$

then we say that f is ***differentiable*** *at the point* x_0.

We also talk about ***linearization*** *of the function f.*

NB: remember that $\circ(x - x_0) = (x - x_0)\,\epsilon(x - x_0)$ with $\epsilon(x - x_0) \xrightarrow[x \to x_0]{} 0$.

Theorem: (differentiability equivalent to differentiability)

*Let be f a function defined on an **open interval** $I \subset \mathbb{R}$ and $x_0 \in I$.*

f is **derivable** at x_0 ***i.e.,*** $f'(x_0) := \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ *exists **if and only if***

f is **differentiable** at x_0 ***i.e.,*** *there is a linear map* $x \in V_{x_0} \to L(x)$ *such as*

$$f(x) = f(x_0) + L(x - x_0) + \circ(x - x_0).$$

We actually have $\quad L(x - x_0) = f'(x_0)\,(x - x_0).$

Indeed, the existence of the limit

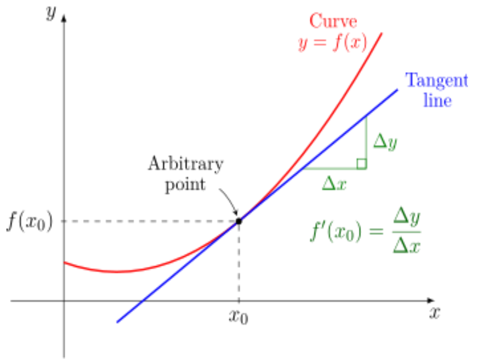
$f'(x_0) := \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ is equivalent to one of the following two writings

$$f(x) = f(x_0) + f'(x_0)\,(x - x_0) + \circ(x - x_0) \quad \text{or} \quad f(x_0+h) = f'(x_0) \times h + f(x_0) \; + \; \circ(h).$$

Theorem: (linearization or approximation of order 1)

*If f is **differentiable (differentiable)** at x_0 then we can approximate $f(x)$ close to x_0.by a linear expression (**approximation of order 1**):*

$$f(x) \approx f(x_0) + f'(x_0)\,(x - x_0)$$



ATTENTION : Linearization depends on the point at which the function is linearized.

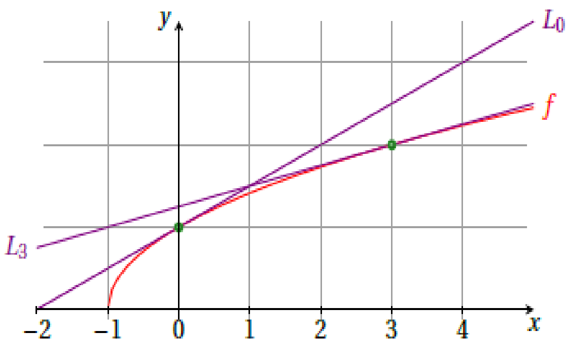
For example, linearizing the function $x \to f(x) = \sqrt{1+x}$ gives

Close to $x_0 = 0$

$$f(x) \approx f(0) + f'(0)\,(x - 0) = 1 + \tfrac{1}{2}x$$

Close to $x_0 = 3$

$$f(x) \approx f(3) + f'(3)\,(x - 3) = \tfrac{5}{4} + \tfrac{1}{4}x.$$



Example.1

Let $f(x) = (1+x)^n$, we have $f'(x) = n\,(1+x)^{n-1}$, linearization $f(x) \approx f(0) + f'(0)\,(x - 0)$, we deduce

$$\boxed{(1+x)^n \approx 1 + n\,x \quad pour\ x << 1}$$

Simple formula to remember. It makes possible to calculate approximations of roots and powers of numbers close to unity. For examples:

$$\sqrt[3]{1.2} = (1+0.2)^{\frac{1}{3}} \approx 1 + \tfrac{1}{3}(0.2) \approx 1.066\ldots \quad \text{(with calculator: } \sqrt[3]{1.2} = 1.062\ldots)$$

$$(1.002)^{100} = (1+0.002)^{100} \approx 1 + 100\,(0.002) = 1.2 \qquad \text{(with calculator: } = (1.002)^{100} = 1.22\ldots)$$

Example.2

Let $f(x) = \sin x$, linearization $\sin(x) \approx \sin(0) + \cos(0)\,(x-0)$, we deduce

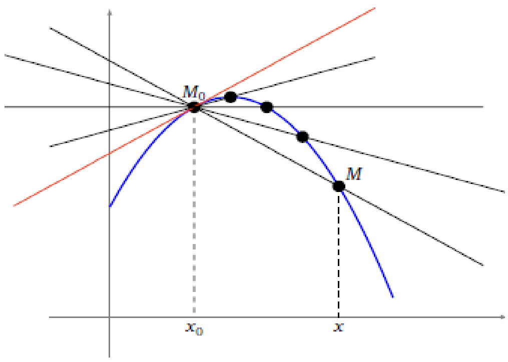
$$\boxed{\sin(x) \approx x \quad pour\ x << 1}$$

This is the linearization that is performed to solve the pendulum equation in physics.

2.2. Line tangent to a point

The ***straight line*** which passes through the distinct points $(x_0, f(x_0))$ and $(x, f(x))$ has as ***slope coefficient*** $\dfrac{f(x)-f(x_0)}{x-x_0}$.

Taking the ***limit***, we find that the ***slope coefficient*** of the ***tangent*** is $f'(x_0)$.
An equation of the tangent at the point $(x_0, f(x_0))$ is then: $y = f'(x_0)\,(x-x_0) + f(x_0)$.

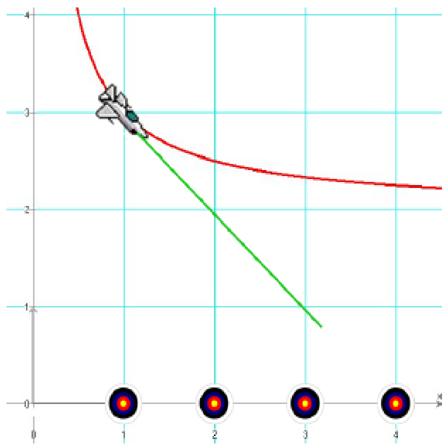


Exercise

The trajectory of an airplane in the opposite figure has the equation $y = \dfrac{2x+1}{x}$ *. The aircraft fires a laser beam along the tangent to its trajectory towards targets placed on the x'Ox axis at abscissa 1, 2, 3 and 4.*

a) *Will target no 4 be hit if the player shoots when the plane is at position (1, 3)?*

b) *Determine the abscissa of the plane allowing to reach the target no 2.*



Correction

a) *Target no 4 will be hit if it is on the tangent to the curve at (1; 3).*

The derivative Is $y' = \dfrac{2\,.\,x-(2x+1)\,.\,1}{x^2} = \dfrac{-1}{x^2}$ *and the tangent equation is*

$y = f'(1)(x-1) + 3 = -x + 4.$

For $x = 4$ *we have* $y = -4 + 4 = 0.$ *Therefore target no. 4 will be affected.*

b) *For target no. 2 to be hit, the tangent at* $(x_0, ?)$ *of the aircraft trajectory must pass through target no. 2; therefore, the couple* $(x;y) = (2;0)$ *must verify the equation*

$y = f'(x_0)(x - x_0) + f(x_0)$ ***i.e.*** $0 = \frac{-1}{x_0^2}(2 - x_0) + \frac{2x_0 + 1}{x_0}$,

that is $0 = \frac{-2 + x_0}{x_0^2} + \frac{2x_0^2 + x_0}{x_0^2}$ *or again* $2x_0^2 + 2x_0 - 2 = 0$. $\Delta' = 5$, $x_1 = \frac{-1 \pm \sqrt{5}}{2}$.

One can deduce the abscissa of the plane-position making possible to reach target no. 2. is $x_0 = \frac{-1 + \sqrt{5}}{2}$.

3 HIGHER-ORDER APPROXIMATION

3.1. Limited Taylor-Young expansion

Definition: (Limited development)

Let $a \in I$ *and* $n \in \mathbb{N}$. *We say that a function f admits a* ***limited expansion*** *(**LE**) to order n, at point a, if there are real numbers* $c_0, c_1, ..., c_n$ *such that for all x close enough to a we have:*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + ... + c_n(x-a)^n + o[(x-a)^n]$$

We recall that $\circ[(x-a)^n] = [(x-a)^n]\,\epsilon(x-a)$ *with* $\epsilon(x-a) \underset{x \to a}{\longrightarrow} 0$.

- ✓ *The term* $c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + ... + c_n(x-a)^n$ *is called the* ***polynomial part*** *of the LE.*
- ✓ *The term* $\circ[(x-a)^n]$ *is the* ***rest*** *of the LE.*
- ✓ *The limited development (**LE**) if it exists is **unique.***
- ✓ *If the function* f *is **even** (resp. **odd**) then the polynomial part of its LE at 0 contains only monomials of **even** (resp. **odd**) **degrees**.*

Theorem: (Taylor-Young formula)

Let f *be a function is of class* C^n *on* I *and* $a \in I$. *then for all* $x \in I$ *we have:*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + o[(x-a)^n]$$

*The **limited expansion** of* $f(x)$ *in the right-hand side of equality is called* ***Taylor-Young polynomials***.

For n=1: we find the approximation of order 1 (***linear***):

$$f(x) \approx f(a) + f'(a)\,(x-a).$$

For n=2: we find the approximation of order 2 (***quadratic***):

$$f(x) \approx f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2.$$

Example

Let's look for various approximations of $f(x) = \exp x$ around the point $a = 0$

✓ **Approximation d'ordre 1 (*linéaire*) :**

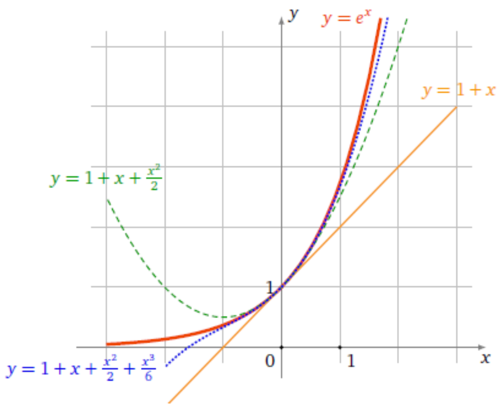
$f(x) \approx e^0 + e^0(x-0) = 1 + x$

✓ **Approximation d'ordre 2 (*quadratique*) :**

$$f(x) \approx e^0 + e^0\,(x-0) + \frac{e^0}{2!}\,(x-0)^2 = 1 + x + \frac{1}{2}x^2$$

✓ **Approximation d'ordre 3 :**

$$f(x) \approx e^0 + e^0(x-0) + \frac{e^0}{2!}(x-0)^2 + \frac{e^0}{3!}(x-0)^3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}$$



Example

For $f(x) = \ln(1+x)$ and $f(x) \approx e^0 + e^0(x-0) = 1 + x$ we have: $f(0) = 0$, $f'(x) = \dfrac{1}{1+x} \implies f'(0) = 1$,
$f''(x) = \dfrac{-1}{(1+x)^2} \implies f'(0) = -1$, $f'''(x) = \dfrac{2}{(1+x)^3} \implies f^3(0) = 2$, hence

✓ ***Linear* approximation (of order 1) :**

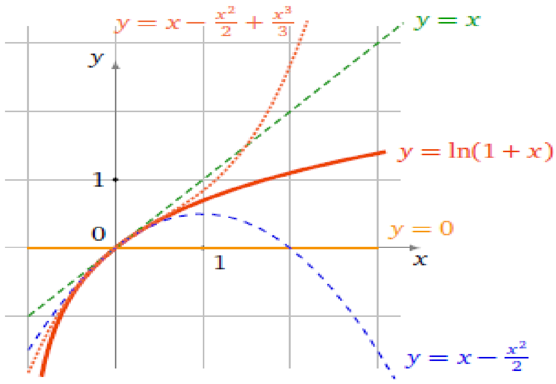
$f(x) \approx 0 + 1\,(x-0) = x$

✓ ***Quadratic* approximation (of order 2) :**

$$f(x) \approx 0 + 1\,(x-0) + \frac{-1}{2!}\,(x-0)^2 = x - \frac{1}{2}x^2$$

✓ **Approximation of order 3 :**

$$f(x) \approx 0 + 1(x-0) + \frac{-1}{2!}(x-0)^2 + \frac{2}{3!}(x-0)^3 = 1 + x - \frac{1}{2}x^2 +$$



Note (important):

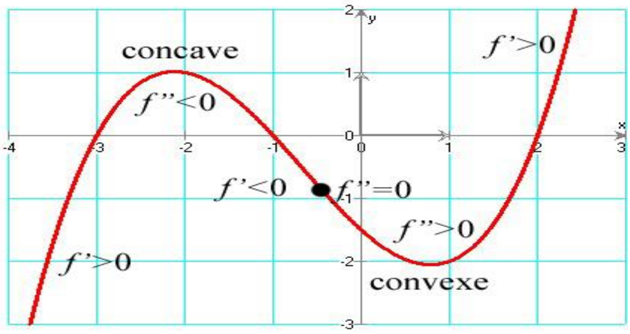
The equation of the tangent at the point of abscissa then a is
$y = f(a) + f'(a)\,(x-a)$. The quadratic approximation (of order 2) makes it possible to study the curvature *of the curve* of the function f

$$f(x) \approx f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 \iff f(x) - y \approx \frac{f''(a)}{2!}\,(x-a)^2.$$

So, on an interval I we have:

- ✓ ***If $f'' < 0$ then the curve of f is below the tangent: concave function.***
- ✓ ***If $f'' > 0$ then the curve of f is above the tangent: function convex.***

The point where there is a change in curvature is called ***the inflection point****. To determine it analytically, it is necessary to solve the equation $f''(x) = 0$and then search among the solutions for those where f'' changes the sign.*



> ***Theorem: (Error of the approximation)***
>
> *If a function f is $n+1$ differentiable and P_n is its Taylor polynomial of order n generated by f at $a \in I$, if $\left|f^{(n+1)}(x)\right|$ is bounded over I by a real M* ***i.e.,*** $\left|f^{(n+1)}(x)\right| \leq M$, *then $\forall x \in I$:*
>
> $$\left|f(x) - P_n(x)\right| \leq \frac{|x-a|^{n+1}}{(n+1)!} M.$$

Example

The linearization close to $a = 0$ of $f(x) = \sin x$ gives $\sin(x) \approx x$.

What is the precision of this approximation if $|x| \leq 0.5$ **i.e.,** $x \in [-0.5, +0.5]$?

We have $\max\limits_{|x|\leq 0.5} |f''(x)| = \max\limits_{|x|\leq 0.5} |-\sin(x)| = \sin(0.5)$ *we deduce*

$$\forall x \in [-0.5, +0.5] \quad : \quad \left|f(x) - P_1(x)\right| = \left|\sin(x) - x\right| \leq \frac{(0.5)^2}{2!} \sin(0.5) < 0.06.$$

4 LE AT THE ORIGIN OF USUAL FUNCTIONS

We have to retain the following LE at 0 of usual functions:

$$\boxed{\exp x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + x^n \epsilon(x)}$$

$$\operatorname{ch} x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + x^{2n+1} \epsilon(x)$$

$$\operatorname{sh} x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + x^{2n+2} \epsilon(x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + x^{2n+1} \epsilon(x)$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + x^{2n+2} \epsilon(x)$$

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + x^n \epsilon(x)}$$

$$\boxed{(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n + x^n \epsilon(x)}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + x^n \epsilon(x)$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + x^n \epsilon(x)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{1}{8} x^2 + \cdots + (-1)^{n-1} \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} x^n + x^n \epsilon(x)$$

Important remarks:

- *The LE of* $\cosh x$ *is the even part of the DL of* $\exp x$ *(we retain the monomials of even degree).*
- *The LE of* $\sinh x$ *is the odd part of the DL of* $\exp x$ *(we retain only the odd degrees).*
- *The LE of* $\cos x$ *is the even part of the DL of* $\exp x$ *by alternating the sign* $+$*and* $-$.
- *The LE* $\sin x$ *is the odd part of* $\exp x$ *by alternating the signs* $+$*and* $-$.
- *For* $\ln(1+x)$ *there is no constant term, no factorial and the signs alternate.*

4.1. LE of functions at any point

The function f admits a LE close to a point $x = a$ **if and only** if the function $t \to f(t+a)$ admits a LE close to $x = 0$.

Therefore, we reduce the problem to 0 by the change of variables $t = x - a$.

Examples.

1. LE of $f(x)=e^x$ at $a=1$.

We pose $t=x-1$. If *x* is close to 1 then *t* is close to 0.

We will look for a LE of e^t near $t=0$.

$$e^x=e^{t+1}=e\,e^t=e\,[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+...+\frac{t^n}{n!}+o(t^n)]=e\,[1+(x-1)+\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+...+\frac{(x-1)^n}{n!}+o[(x-1)^n]]$$

So close to $a=1$ we get

$$e^x=e\,[1+(x-1)+\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+...+\frac{(x-1)^n}{n!}+o[(x-1)^n]].$$

2. LE of $g(x)=\sin x$ close to $a=\pi/2$. We pose $t=x-\pi/2$, we have $x\to\pi/2 \iff t\to 0$.

$$\sin x=\sin(t+\pi/2)=\cos(t)=1-\frac{t^2}{2!}+\frac{t^4}{4!}-...+(-1)^n\frac{t^{2n}}{(2n)!}+o(t^{2n+1})$$
$$=1-\frac{(x-\pi/2)^2}{2!}+\frac{(x-\pi/2)^4}{4!}-...+(-1)^n\frac{(x-\pi/2)^{2n}}{(2n)!}+o[(x-\pi/2)^{2n+1}]$$

3. LE of $h(x)=\ln(1+3x)$ at a=1 to order 3.

We set $t=x-1$, we have $x\to 1 \iff t\to 0$.

$$\ln(1+3x)=\ln(1+3(t+1))=\ln(4+3t)=\ln 4(1+\frac{3t}{4})=\ln 4+\ln(1+\frac{3t}{4}).$$

We pose $T=\frac{3t}{4}$, we have $x\to 1 \iff t\to 0 \iff T\to 0$; we use

$$\ln(1+T)=T-\tfrac{T^2}{2}+...+(-1)^{n-1}\tfrac{T^n}{n}+o(T^n).$$

$$h(x)=\ln(1+3x)=\ln 4+\ln(1+\frac{3t}{4})=\ln 4+\ln(1+T)=\ln 4+T-\frac{T^2}{2}+\frac{T^3}{3}+o(T^3)$$
$$=\ln 4+\frac{3t}{4}-\frac{\left(\frac{3t}{4}\right)^2}{2}+\frac{\left(\frac{3t}{4}\right)^3}{3}+o((\frac{3t}{4})^3)=\ln 4+\frac{3}{4}t-\frac{9}{32}t^2+\frac{9}{64}t^3+o[t^3]$$
$$=\ln 4+\frac{3}{4}(x-1)-\frac{9}{32}(x-1)^2+\frac{9}{64}(x-1)^3+o[(x-1)^3]\ .$$

4.2. Operations on limited developments

Let *f* and *g* be two functions which admit LEs at 0 to order *n:*

$$f(x)=c_0+c_1\,x+...+c_n\,x^n+o(x^n):=P_f+o(x^n),$$
$$g(x)=d_0+d_1\,x+...+d_n\,x^n+o(x^n):=P_g+o(x^n).$$

Theorem: (Sum and product)

- *The TAYLOR polynomial of order n generated for the* ***sum*** $f+g$ *is the* ***polynomial sum*** $P_f + P_g$*;*
  $$(f+g)(x) = (c_0+d_0) + (c_1+d_1)\,x + \dots + (c_n+d_n)\,x_n + o(x^n).$$
- *The TAYLOR polynomial of order n generated for the* ***product*** $f.g$ *is the* ***polynomial product*** $P_f \times P_g$ ***truncated*** *to order n,* ***i.e.,*** *that we keep only the monomials of degree ≤ n;*

Example.

We have the LE of order 2:

$\cos x = 1 - \frac{1}{2}x^2 + o(x^2)$ and $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)$ then:

$$\cos x + \sqrt{1+x} = [1 - \frac{1}{2}x^2 + o(x^2)] + [1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)]$$
$$= (1+1) + \frac{1}{2}x + (-\frac{1}{2} - \frac{1}{8})\,x^2 + o(x^2) \;=\; 2 + \frac{1}{2}x - \frac{5}{8}x^2 + o(x^2)$$

$$\cos x\;\sqrt{1+x} = [1 - \frac{1}{2}x^2 + o(x^2)] \times [1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)]$$
$$= 1 \times [1 + \frac{1}{2}x - \frac{1}{8}x^2] - \frac{1}{2}x^2 \times 1 + o(x^2) + o(x^2) \;=\; 1 + \frac{1}{2}x - \frac{5}{8}x^2 + o(x^2)$$

Theorem: (Composition)

- *If* $g(0)=0$ *then the* ***composite*** *function* $f \circ g$ *admits a LE of order n at a=0 whose polynomial part is the truncated polynomial at order n of the composite* $P_f[P_g(x)]$*.*

Examples:

1) Calculation of the LE of $h(x) = \sin \ln(1+x)$ at 0 to order 3.

We put here $f(u) = \sin u$ and $u = g(x) = \ln(1+x)$. We have

$$(f \circ g)(x) = f[g(x)] = f(u) = \sin u = \sin \ln(1+x) \qquad \text{and } g(0)=0.$$

The LEs: $\sin u = u - \frac{1}{3!}u^3 + o(u^3)$ and $u = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$,

so $u^2 = [x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)]^2 = x^2 - 2\,x\,\frac{1}{2}x^2 + o(x^3) = x^2 - x^3 + o(x^3)$

and $u^3 = [x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)] \times [x^2 - x^3 + o(x^3)] = x^3 + o(x^3)$.

Consequently

$$\sin\ln(1+x)=\sin u=u-\frac{1}{3!}u^3+o(u^3)\;=\;[x-\frac{1}{2}x^2+\frac{1}{3}x^3+o(x^3)]-\frac{1}{6}[x^3+o(x^3)]+o(x^3)$$
$$=x-\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3)$$

2) Calculation of the LE of $h(x)=\sqrt{\cos x}$ near 0 to order 4.

We know the LEs: $\cos x=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+o(u^4)$ and $\sqrt{1+u}=1+\frac{1}{2}u-\frac{1}{8}u^2+o(x^3)$

We put $f(u)=\sqrt{u}$ and $g(x)=1+u=\cos(x)$. We have

$$h(x)=(f\circ g)(x)=f[g(x)]=f(1+u)=\sqrt{1+u} \qquad \text{and} \qquad g(0)=0.$$

$u=\cos x-1=[1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+o(u^4)]-1=-\frac{1}{2}x^2+\frac{1}{24}x^4+o(u^4)$

$u^2=[-\frac{1}{2}x^2+\frac{1}{24}x^4+o(u^4)]^2=\frac{1}{4}x^4+o(x^4).$

We deduce

$$\sqrt{\cos x}=\sqrt{1+u}=1+\frac{1}{2}u-\frac{1}{8}u^2+o(x^4)\;=\;1+\frac{1}{2}[-\frac{1}{2}x^2+\frac{1}{24}x^4+o(u^4)]-\frac{1}{8}[\frac{1}{4}x^4+o(x^4)]+o(x^4)$$
$$=1-\frac{1}{4}x^2+\frac{1}{48}x^4-\frac{1}{32}x^4+o(x^4)=1-\frac{1}{4}x^2-\frac{1}{96}x^4+o(x^4)$$

Theorem: (Division)

- *By carrying out the **division according to the <u>increasing</u> powers** of P_f by P_g to the order n we will obtain the writing:*

$$P_f=P_g\;Q+x^{n+1}\,R \qquad \text{with } \deg Q\le n.$$

*Then **Q is the polynomial part of the LE** at 0 to order n of* $\dfrac{P(x)}{Q(x)}$.

Example

Find the LE of $\dfrac{2+x+2x^2}{1+x^2}$ to order 2.

$$\begin{array}{l|l}
2+x+2x^3 & 1+x^2 \\ \cline{2-2}
2+2x^2 & 2+x-2x^2 \\ \cline{1-1}
x-2x^2+2x^3 & \\
x+x^3 & \\ \cline{1-1}
-2x^2+x^3 & \\
-2x^2-2x^4 & \\ \cline{1-1}
x^3+2x^4 &
\end{array}$$

From the Euclidian division we deduce

$$\frac{2+x+2x^2}{1+x^2}=2+x-2x^2+o(x^2)$$

4.3. Applications of LEs

4.3.1. Limit calculations:

1) Calculate $\lim_{x\to 0}\dfrac{\cos x-1}{e^x-1}$. Let's use the LEs:

$$e^x = 1 + x + \frac{t^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + o(x^n)\ ;$$

$$\cos x = 1 - \frac{t^2}{2!} + \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots + \frac{x^n}{n!} + o(x^{2n}).$$

$$\frac{\cos x-1}{e^x-1} = \frac{1-\frac{1}{2}x^2+o(x^2)-1}{1+x-\frac{1}{2}x^2+o(x^2)-1} = \frac{-\frac{1}{2}x^2+o(x^2)}{x-\frac{1}{2}x^2+o(x^2)} = \frac{-x^2+o(x^2)}{2x-x^2+o(x^2)}.$$

$$\begin{array}{l|l}
-x^2 & 2x-x^2 \\ \cline{2-2}
-x^2+\frac{1}{2}x^3 & -\frac{1}{2}x \\ \hline
-\frac{1}{2}x^3 &
\end{array}$$

We deduce $\qquad \lim_{x\to 0}\dfrac{\cos x-1}{e^x-1} = \lim_{x\to 0} -\dfrac{1}{2}x + o(x) = 0.$

2) Calculate $\lim_{x\to 0}\dfrac{\ln(1+x)-\tan x+\frac{1}{2}\sin^2 x}{3x^2\, sin^2 x} := \dfrac{f(x)}{g(x)}.$ $\qquad$ We recall:

$$\ln(1+x) = x - \frac{t^2}{2} + \ldots + (-1)^{n-1}\frac{x^n}{n} + o(x^n)\ ;$$

$$\sin x = x - \frac{x^3}{3!} + \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$$

$$\cos x = 1 - \frac{x^2}{2!} + \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + o(x^{2n});\ \tan x = \frac{\sin x}{\cos x} = \frac{x-\frac{x^3}{3!}+\ldots}{1-\frac{x^2}{2!}+\ldots}.$$

$$\begin{array}{l|l}
x-\frac{1}{6}x^3+\ldots & 1-\frac{1}{2}x^2 \\ \cline{2-2}
x-\frac{1}{2}x^3 & x+\frac{1}{3}x^3+\ldots \\ \hline
+\frac{1}{3}x^3+\ldots & \\
+\frac{1}{3}x^3-\frac{1}{6}x^5 & \\ \hline
\ldots &
\end{array}$$

We deduce

$$f(x) := \ln(1+x) - \tan x + \frac{1}{2}\sin^2 x$$

$$= [x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + o(x^4)] - [x + \frac{1}{3}x^3 + o(x^4)] + \frac{1}{2}[x - \frac{1}{3!}x^3 + o(x^4)]^2$$

$$= -\frac{1}{2}x^2 - \frac{1}{4}x^4 + o(x^4) + \frac{1}{2}[x^2 - 2\,x\,\frac{1}{3!}x^3 + o(x^4)]$$

$$= -\frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{2}x^2 - \frac{1}{6}x^4 + o(x^4) = -\frac{5}{12}x^4 + o(x^4)\ .$$

$$g(x) := 3x^2\ \sin^2 x\ =\ 3x^2\,[x - \frac{1}{3!}x^3 + o(x^4)]^2$$

$$= 3x^2\,[x^2 - 2\,x\,\frac{1}{3!}x^3 + o(x^4)]\ =\ 3x^4 + o(x^4)\ .$$

Then $\lim\limits_{x\to 0}\dfrac{\ln(1+x) - \tan x + \frac{1}{2}\sin^2 x}{3x^2\,sin^2 x} = \lim\limits_{x\to 0}\dfrac{-\frac{5}{12}x^4 + o(x^4)}{3x^4 + o(x^4)} = -\dfrac{5}{36}$.

NB: by calculating the LE at a lower order, we could not have concluded, because we would have obtained $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0}\dfrac{o(x^4)}{o(x^4)}$ which remains an indeterminate form.

4.3.2. Equivalences:

1) Give simple equivalents close to 0 for the following functions:

a) $2e^x - \sqrt{1+4x} - \sqrt{1+6x^2}$ **b)** $(\cos x)^{\sin x} - (\cos x)^{\tan x}$.

a) $2e^x - \sqrt{1+4x} - \sqrt{1+6x^2} := f(x)$. We have the LEs:

$e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots + \dfrac{x^n}{n!} + o(x^n)$ and

$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{2!}x^3 + \ldots + \frac{\alpha(\alpha-1)..(\alpha-n+1)}{n!}x^n + o(x^n)$,

then for $\alpha = \frac{1}{2}$

$$(1+x)^{\frac{1}{2}} = \sqrt{1+x} = 1 + \tfrac{1}{2}x + \tfrac{1(-1)}{2^2\ 2!}x^2 + \tfrac{1(-1)(-3)}{2^3\ 3!}x^3 + \ldots + (-1)^{n-1}\tfrac{1(1)(3)(5)..(2n-3)}{2^n\ n!}x^n + o(x^n).$$

To order 3 we'll have $(1+x)^{\frac{1}{2}} = \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + +o(x^3)$.

We deduce

$$f(x) = [2 + 2x + x^2 + \frac{x^3}{3} + o(x^3)] - [1 + \frac{1}{2}(4x) - \frac{1}{8}(4x)^2 + \frac{1}{16}(4x)^3 + o(x^3)] - [1 + \frac{1}{2}(6x^2) + o(x^3)]$$

$$= +x^2 + \frac{x^3}{3} + \frac{16}{8}x^2 - \frac{4^3}{16}x^3 - \frac{6}{2}x^2 + o(x^3) = +\frac{x^3}{3} - 4\,x^3 + o(x^3) = -\frac{11}{3}x^3 + o(x^3)$$

So close to 0 we have $2e^x - \sqrt{1+4x} - \sqrt{1+6x^2} \sim -\dfrac{11}{3}x^3$.

b) $g(x) := (\cos x)^{\sin x} - (\cos x)^{\tan x} = \exp[\sin x \ \ln\cos x] - \exp[\tan x \ \ln\cos x]$.

We know the LEs: $\sin x = x - \dfrac{x^3}{3!} + ... + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$

$\cos x = 1 - \dfrac{x^2}{2!} + ... + (-1)^n \dfrac{x^{2n}}{(2n)!} + o(x^{2n})$; $\tan x = \dfrac{\sin x}{\cos x} = x + \frac{1}{3}\, x^3 + o(x^4)$

$\ln(1-x) = x + \frac{1}{2}\, x^2 + ... + \frac{1}{n}\, x^n + o(x^n)$ and $e^x = 1 + x + \dfrac{t^2}{2!} + \dfrac{x^3}{3!} + ... + \dfrac{x^n}{n!} + o(x^n)$

.

We deduce

$$\begin{aligned}
g(x) &= \exp[(\sin x)\ \ln(\cos x)] - \exp[(\tan x)\ \ln(\cos x)] \\
&= \exp\left[\left(x - \frac{1}{3!}\, x^3 + o(x^3)\right)\ \ln\left(1 - \frac{x^2}{2!} + o(x^3)\right)\right] - \exp\left[\left(x + \frac{1}{3}\, x^3 + o(x^3)\right)\ \ln\left(1 - \frac{x^2}{2!} + o(x^3)\right)\right] \\
&= \exp\left[\left(x - \frac{1}{3!}\, x^3 + o(x^3)\right)\left(\frac{x^2}{2} + \frac{1}{2}[\frac{x^2}{2}]^2 + o(x^3)\right)\right] - \exp\left[\left(x + \frac{1}{3}\, x^3 + o(x^3)\right)\left(\frac{x^2}{2} + \frac{1}{2}[\frac{x^2}{2}]^2 + o(x^3)\right)\right] \\
&= \exp\left[\frac{x^3}{2} + \frac{1}{2}\frac{x^5}{4} - \frac{1}{3!}\,\frac{x^5}{2} + o(x^5)\right] - \exp\left[\frac{x^3}{2} + \frac{1}{2}\frac{x^5}{4} + \frac{1}{3}\,\frac{x^5}{2} + o(x^5)\right] \\
&= 1 + \frac{x^3}{2} + \frac{1}{2}\frac{x^5}{4} - \frac{1}{6}\,\frac{x^5}{2} + o(x^5) - 1 - \frac{x^3}{2} - \frac{1}{2}\frac{x^5}{4} - \frac{1}{3}\,\frac{x^5}{2} + o(x^5) \\
&= -\frac{3}{6}\,\frac{1}{2}\, x^5 + o(x^5) = -\frac{1}{4}\, x^5 + o(x^5)
\end{aligned}$$

So close to 0 we have $\quad (\cos x)^{\sin x} - (\cos x)^{\tan x} \sim -\dfrac{1}{4}\, x^5$.

2) Give an equivalent close to $+\infty$ of $\sqrt{x^2+1} - 2\sqrt[3]{x^3+x} + \sqrt[4]{x^4+x^2}$.

Reminder: $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}\, x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{2!}\, x^3 + ... + \frac{\alpha(\alpha-1)..(\alpha-n+1)}{n!}\, x^n + o(x^n)$

then close to 0 we have

$(1+x)^{\frac{1}{2}} = \sqrt{1+x} = 1 + \frac{1}{2}\, x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\, x^2 + o(x^2) = 1 + \frac{1}{2}\, x - \frac{1}{2^2\ 2!}\, x^2 + o(x^2)$

$(1+x)^{\frac{1}{3}} = 1 + \frac{1}{3}x + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}\, x^2 + o(x^2) = 1 + \frac{1}{3}x - \frac{2}{3^2\ 2!}\, x^2 + o(x^2)$

$(1+x)^{\frac{1}{4}} = 1 + \frac{1}{4}\, x + \frac{\frac{1}{4}(\frac{1}{4}-1)}{2!}\, x^2 + o(x^2) = 1 + \frac{1}{4}\, x - \frac{3}{4^2\ 2!}\, x^2 + o(x^2)$

Noticing that $x \to +\infty \iff \dfrac{1}{x} \to 0$, we deduce that close to $+\infty$ we have

$\sqrt{x^2+1} = x\ \sqrt{1 + \left(\frac{1}{x^2}\right)} = x\ \left[1 + \frac{1}{2}\frac{1}{x^2} - \frac{1}{8}\left(\frac{1}{x^2}\right)^2 + o(\frac{1}{x^4})\right] = x + \frac{1}{2}\,\frac{1}{x} - \frac{1}{8}\,\frac{1}{x^3} + o(\frac{1}{x^4})$

$-2\sqrt[3]{x^3+x} = -2x\ \sqrt[3]{1 + \left(\frac{1}{x^2}\right)} = -2x\ \left[1 + \frac{1}{3}\left(\frac{1}{x^2}\right) - \frac{1}{9}\left(\frac{1}{x^2}\right)^2 + o(\frac{1}{x^4})\right] = -2x - \frac{2}{3}\,\frac{1}{x} + \frac{2}{9}\,\frac{1}{x^3} + o(\frac{1}{x^4})$

$\sqrt[4]{x^4+x^2} = x\ \sqrt[4]{1 + \left(\frac{1}{x^2}\right)} = x\ \left[1 + \frac{1}{4}\frac{1}{x^2} - \frac{3}{32}\left(\frac{1}{x^2}\right)^2 + o(\frac{1}{x^4})\right] = x + \frac{1}{4}\,\frac{1}{x} - \frac{3}{32}\,\frac{1}{x^3} + o(\frac{1}{x^4})$

Adding these results, we get

$$f(x) := \sqrt{x^2+1} - 2\sqrt[3]{x^3+x} + \sqrt[4]{x^4+x^2}$$
$$= [x + \frac{1}{2}\frac{1}{x} - \frac{1}{8}\frac{1}{x^3} + o(\frac{1}{x^4})] + [-2x - \frac{2}{3}\frac{1}{x} + \frac{2}{9}\frac{1}{x^3} + o(\frac{1}{x^4})] + [x + \frac{1}{4}\frac{1}{x} - \frac{3}{32}\frac{1}{x^3} + o(\frac{1}{x^4})]$$
$$= \frac{1}{12}\frac{1}{x} + o(\frac{1}{x})$$

So close to $+\infty$ $\qquad \sqrt{x^2+1} - 2\sqrt[3]{x^3+x} + \sqrt[4]{x^4+x^2} \sim \frac{1}{12}\frac{1}{x}$.

4.3.3. Others:

Find the tangent of the graph, at point of abscissa $a = 1/2$, of a function f *defined by* $f(x) = x^4 - 2x^3 + 1$; and specify the position of the graph with respect to the tangent

Let's use the LE of $f(x)$ at point $a = \frac{1}{2}$. $f'(x) = 4x^3 - 6x^2$, $f''(x) = 12x^2 - 12x$; then

$$f(x) = f(1/2) + f'(1/2)(x-1/2) + \frac{f'(1/2)}{2!}(x-1/2)^2 + o[(x-1/2)^2] = \frac{13}{36} - (x-1/2) - \frac{3}{2}(x-1/2)^2 + o[(x-1/2)^2]$$

We deduce the equation of the tangent $y = \frac{13}{36} - (x - 1/2)$.

The position of the graph with respect to the tangent depends on the sign of

$$f(x) - y = -\frac{3}{2}(x-1/2)^2 + o[(x-1/2)^2]$$

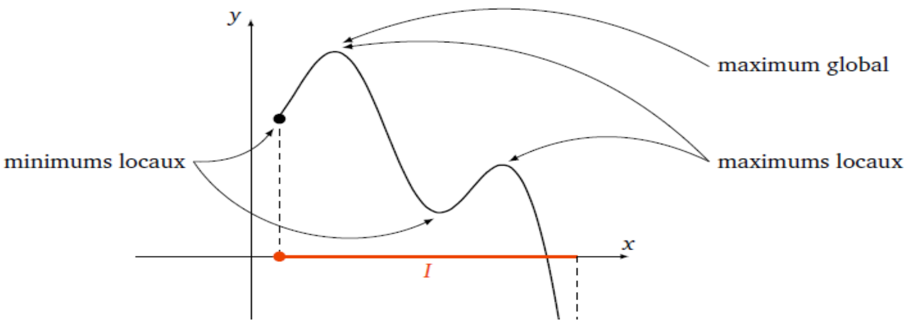
which is negative; this means that the graph is below the tangent.

5 OPTIMIZATION (LOCATION AND NATURE OF EXTREMA)

Definitions

Let $f : [a,b] \to \mathbb{R}$ be a function. We say that

- ✓ f is **bounded** *in [a,b] if there exists a real $M \geq 0$ such as*
  $\forall x \in [a,b] \ : \ |f(x)| \leq M \quad (i.e. \ -M \leq f(x) \leq +M)$;
- ✓ f *admits a* **global maximum** *(resp.* **minimum***) at $c \in [a,b]$ if*
  $\forall x \in [a,b] \ : \ f(x) \leq f(c)$ *(resp.* $\forall x \in [a,b] \ : \ f(x) \geq f(c)$*);*
- ✓ f *admits at $c \in [a,b]$ a* **local** *(or relative)* **maximum** *(resp.* **minimum***) if there exists a neighbourhood $Vois$ of c such that*
  $\forall x \in Vois(c) \ : \ f(x) \leq f(c)$ *(resp.* $\forall x \in Vois(c) \ : \ f(x) \geq f(c)$*)*.



Critical (or stationary) point

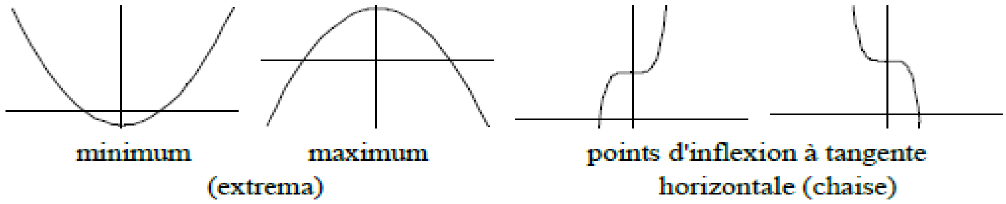
Let $f\ :\ [a,b] \to \mathbb{R}$ be a differentiable function at $c \in [a,b]$.
*If $f'(c) = 0$ then c is called **critical point** (or **stationary point**).*

Proposition (extremum implies critical point)

Let $f\ :\ [a,b] \to \mathbb{R}$ be a differentiable function in $c \in [a,b]$.
If f has a local extremum at c, then c is a critical point ($f'(c) = 0$).

NB: If $f' = 0$ at a point, then there are two possibilities for this point

- ✓ It is an **extremum** of the function or
- ✓ It is an **inflection point with horizontal tangent**.



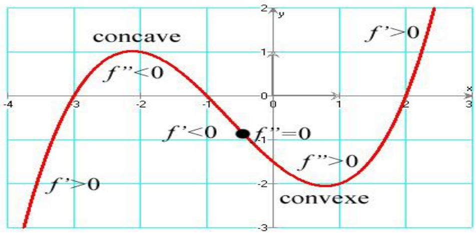
Important: The extrema of a function are to be found among the critical (stationary) points.

Proposition (second derivative and classification of extrema)

*Let $f\ :\ [a,b] \to \mathbb{R}$ be a **differentiable** function **at a critical point***
$c \in]a,b[$
($f'(c) = 0$) then:

*1) If $f''(c) < 0$ (concave curve), the function admits a local **maximum** at c,*

*2) If $f''(c) > 0$ (convex curve), the function admits a local **minimum** at c.*



Example

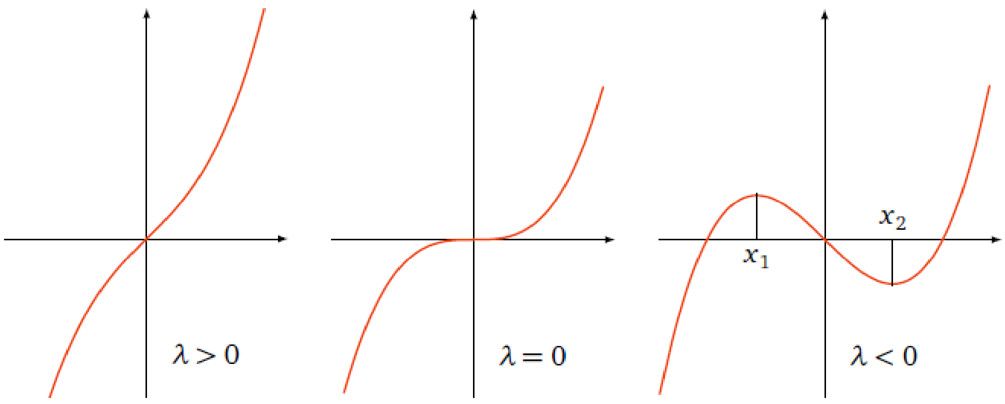
Find the extrema of the function f_λ, with a real-parameter λ, defined by

$$f_\lambda(x) = x^3 + \lambda\, x.$$

The extrema of the function f_λ are to be found among the critical points. The derivative is $f'_\lambda(x) = 3\,x^2 + \lambda$. If *c* is a local extremum, then we'll have

$$f'_\lambda(c) = 3\,c^2 + \lambda = 0.$$

- ✓ If $\lambda > 0$ then $\forall x \in \mathbb{R}\ :\ f'_\lambda(x) > 0$, there are no critical points and so no extremums.
- ✓ If $\lambda = 0$ then $f'_\lambda(x) = x^2 = 0 \iff x = 0$. The second derivative $f''_\lambda(x) = 6\,x$ vanishes at $x = 0$ and changes sign. Therefore $x = 0$ is not an extremum but a point of inflection (*change of curvature*).
- ✓ If $\lambda < 0$ then $f'_\lambda(x) = x^2 - |\lambda| = 0 \iff x = \pm\sqrt{|\lambda|}$. There are two critical points $x = -\sqrt{|\lambda|}$ and $x = +\sqrt{|\lambda|}$. We have $f''_\lambda(x) = 6\,x$.
  - $f''_\lambda(-\sqrt{|\lambda|}) < 0$ then $x = -\sqrt{|\lambda|}$ is a ***local maximum***
  - $f''_\lambda(+\sqrt{|\lambda|}) > 0$ then $x = +\sqrt{|\lambda|}$ is a ***local minimum***.

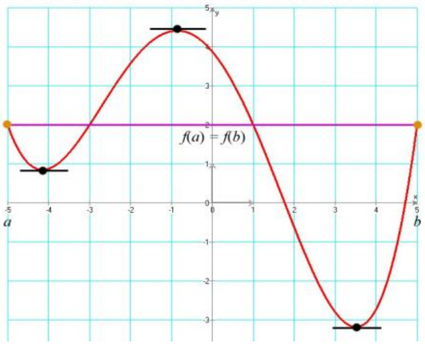


Théorème de Rolle

Let $f : [a,b] \to \mathbb{R}$ be a function such that:

1. *f is continuous on [a; b]*
2. *f is differentiable on]a; b[*
3. $f(a) = f(b)$

then it exists $c \in]a;b[$ such that $f'(c) = 0$
(horizontal tangent).



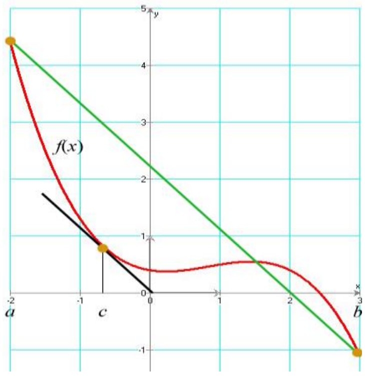
Théorème des accroissements finis

Let $f : [a,b] \to \mathbb{R}$ be a function such that:

1. *f is continuous on [a; b]*
2. *f is differentiable on]a; b[*

then it exists $c \in]a;b[$ such that:

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



Corollary

Let $f : [a,b] \to \mathbb{R}$ be a continuous function on [a, b] and differentiable on]a , b[.

1. $\forall x \in]a,b[: f'(x) \geq 0 \left(f'(x) > 0\right) \implies$ *f is (strictly) increasing;*

2. $\forall x \in]a,b[: f'(x) \leq 0 \left(f'(x) < 0\right) \implies$ *f is decreasing (strictly);*

3. $\forall x \in]a,b[: f'(x) = 0 \implies$ *f is constant.*

Cauchy's theorem " generalized finite increments".

Let f and g be two continuous functions on $[a,b]$, differentiable on $]a,b[$. Suppose that $g(a) \neq g(b)$ and that g' does not vanish on $]a,b[$; then there exists $c \in]a;b[$ such that:

$$\frac{f(b) - f(a)}{b - a} = \frac{f'(c)}{g'(c)}.$$

Hospital Rule (IF $\frac{0}{0}$, $\frac{\infty}{\infty}$)

Let f and g be two functions such that:

1. $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$,

2. *f and g are differentiable nearby $a \in \mathbb{R}$,*

3. *the derived g' does not vanish close to a,*

4. $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ *exists.*

Then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Important Note

The rule is also valid if $a = \pm\infty$, or if $\lim\limits_{x \to a} f(x) = \pm\infty$ and $\lim\limits_{x \to a} g(x) = \pm\infty$.

Examples

1. Calculate $\lim\limits_{x \to 1} \dfrac{\ln(x^2 + x - 1)}{\ln(x)}$. We have an indeterminate form $\frac{0}{0}$.

Let's use the Hospital's rule: set $N(x) = \ln(x^2 + x - 1)$ and $D(x) = \ln(x)$.

$$\lim_{x \to 1} \frac{N'(x)}{D'(x)} = \lim_{x \to 1} \frac{\frac{2x+1}{x^2+x-1}}{\frac{1}{x}} = \lim_{x \to 1} \frac{2x+1}{x^2+x-1} \times \frac{1}{x} = 3$$

We deduce $$\lim_{x \to 1} \frac{\ln(x^2 + x - 1)}{\ln(x)} = 3.$$

2. Calculate the limit $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x}$**.** We apply the HOSPITAL theorem

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1$$

3. Calculate the limit $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$**.**

Let's pose $N(x) = e^x - e^{-x} - 2x$ and $D(x) = x - \sin x$.

$N'(x) = e^x + e^{-x} - 2 \to 0$ and $D'(x) = 1 - \cos x \to 0$. $\lim\limits_{x \to 0} \dfrac{N'(x)}{D'(x)}$ remains IF.

$N''(x) = e^x - e^{-x} \to 0$ and $D''(x) = \sin x \to 0$. $\lim_{x\to 0} \frac{N''(x)}{D''(x)}$ remains IF.

$N'''(x) = e^x + e^{-x} \to 2$ and $D'''(x) = \cos x \to 1$. $\lim_{x\to 0} \frac{N'''(x)}{D'''(x)} = 2$.

We deduce $$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = 2.$$

Exercise

Calculate the following limits: **1)** $\lim_{x\to\pi/2} (x - \pi/2)\,\tan x$ **2)** $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

3) $\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{(x-1)^2}\right)$ **4)** $\lim_{x\to 1} \frac{\sin(\pi\, x)}{\ln x}$.

Correction

1) $\lim_{x\to\pi/2} (x - \pi/2)\,\tan x = \lim_{x\to\pi/2} \frac{x - \pi/2}{\frac{\cos x}{\sin x}} := \lim_{x\to\pi/2} \frac{N(x)}{D(x)}$.

$$\lim_{x\to\pi/2} \frac{N'(x)}{D'(x)} \lim_{x\to\pi/2} = \lim_{x\to\pi/2} \frac{1}{\frac{-1}{\sin^2 x}} = \lim_{x\to\pi/2} \sin^2 x = 0$$

We conclude $$\lim_{x\to\pi/2} (x - \pi/2)\,\tan x = 0$$

2) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{\sin x - x}{x\,\sin x} = \lim_{x\to 0} \frac{N(x)}{D(x)}$

$$\lim_{x\to 0} \frac{N'(x)}{D'(x)} = \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\,\cos x}$$

$$\lim_{x\to 0} \frac{N''(x)}{D''(x)} = \lim_{x\to 0} \frac{-\sin x}{\cos x + \cos x - x\,\sin x} = 0$$

so $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = 0$

3) $\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{(x-1)^2}\right) = \lim_{x\to 1} \frac{(x-1)^2 - \ln x}{(x-1)^2\,\ln x} = \lim_{x\to 1} \frac{N(x)}{D(x)}$

$$\lim_{x\to 1} \frac{N'(x)}{D'(x)} = \lim_{x\to 1} \frac{2(x-1) - \frac{1}{x}}{2(x-1)\,\ln x + \frac{(x-1)^2}{x}} = \infty$$

so $\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{(x-1)^2}\right) = \infty$

4) $\lim_{x\to 1} \frac{\sin(\pi\, x)}{\ln x} = \lim_{x\to 1} \frac{N(x)}{D(x)}$

$$\lim_{x\to 1} \frac{N'(x)}{D'(x)} = \lim_{x\to 1} \frac{\pi\,\cos(\pi\, x)}{\frac{1}{x}} = -\pi$$

so $\lim_{x\to 1} \frac{\sin(\pi\, x)}{\ln x} = -\pi$

7 NUMERICAL METHOD FOR FINDING ZEROS (NEWTON'S METHOD):

We have already seen in the previous chapter the ***method of bisection or dichotomy*** for the approximate calculation of the zeros of a continuous function. This method still works but is not very fast.

We present here a ***faster*** method, ***Newton's method***.

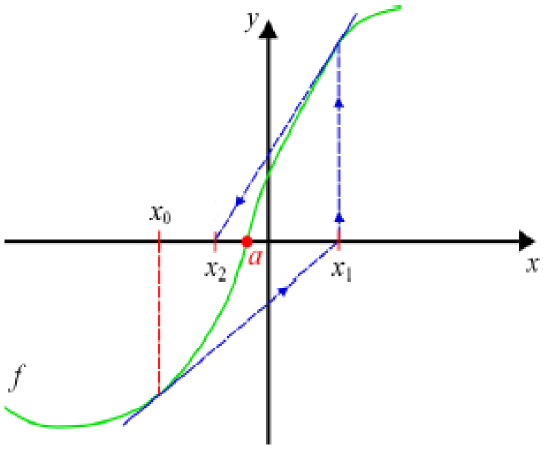
The principle is as follows: given $f : \mathbb{R} \to \mathbb{R}$ a function of class C^1 and a a single zero of f, **i.e.,** $f(a) = 0$ and $f'(a) \neq 0$.

knowing a value x_k close to a, we calculate x_{k+1} by taking the abscissa of the intersection of x-axis with the tangent to the graph of f passing through the point $(x_k, f(x_k))$:

$$\begin{cases} y = f'(x)\,(x - x_k) + f(x_k) \\ y = 0 \end{cases} \iff x = x_k - \frac{f(x_k)}{f'(x_k)}.$$

We thus define the recurrent sequence

$$\begin{cases} x_0 \text{ donnée} \\ x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)} \end{cases} .$$



As a stopping criterion, we can choose to stop when the iterates are close to each other or when the value taken by the function is sufficiently close to zero.

Example *(approximate calculation of $\sqrt{2}$)*

We have $x^2 - 2 = 0 \iff x = \pm\sqrt{2}$. Let's find the zero of the function $f(x) = x^2 - 2$ in the interval [0, 2] with a precision of 10^{-5}.

Using Excel, we defined $x_0 = 1$ and $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)} = x_k - \dfrac{x_k^2 - 2}{2\,x_k}$.

k	xk	| xk - racine(2) |
0	1,000000	0,414214
1	1,500000	0,085786
2	1,416667	0,002453
3	1,414216	0,000002

Exercise

Let's look for a zero of the function $f(x) = \cos x - x^3$.

Correction

$f(0) = 1$ and $f(1) \approx -0.46$, f being continuous we deduce that the zero is between 0 and 1. Let us take as starting value $x_0 = 0.5$. The derivative is $f'(x) = -\sin x - 3x^2$.

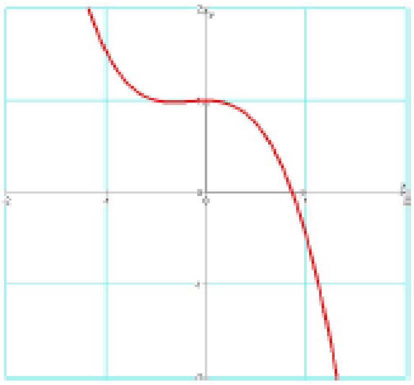
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{\cos(0.5) - 0.5^3}{-\sin(0.5) - 3 \cdot 0.5^2} = 1.11214$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.11214 - \frac{\cos(1.11214) - 1.11214^3}{-\sin(1.11214) - 3 \cdot 1.11214^2} = 0.90967$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.90967 - \frac{\cos(0.90967) - 0.90967^3}{-\sin(0.90967) - 3 \cdot 0.90967^2} = 0.86626$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.86626 - \frac{\cos(0.86626) - 0.86626^3}{-\sin(0.86626) - 3 \cdot 0.86626^2} = 0.86547$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 0.86547 - \frac{\cos(0.86547) - 0.86547^3}{-\sin(0.86547) - 3 \cdot 0.86547^2} = 0.86547$$



$f(x) = \cos(x) - x^3$

6th chap. : Usual Functions

1 POWER FUNCTIONS, SECOND DEGREE POLYNOMIAL

1.1. Power functions

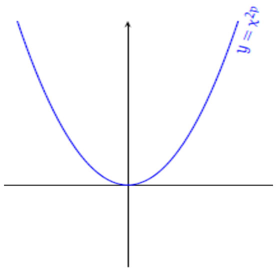
The **power function** *is defined on* $\mathbb{R}$ *by*

$$\forall x \in \mathbb{R} \ : \ f(x) = x^n \ , \ n \in \mathbb{N}^*.$$

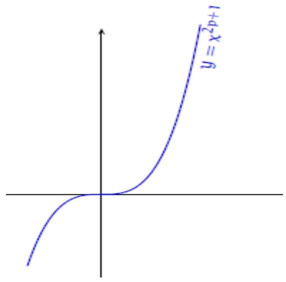
- *It is continuous and indefinitely differentiable, we have:*

$$(x^n)' = n\,x^{n-1} \qquad \textit{And} \quad (x^n)'' = n(n-1)\,x^{n-2} \ ...$$

- *The variation of* x^n *depends on the parity of* $n \in \mathbb{N}^*$.



$n = 2p, \ p \in \mathbb{N}^*$



$n = 2p + 1, \ p \in \mathbb{N}^*$

- *When* $n \in \mathbb{N}^*$ *is* ***even*** *the curve is a* **parabola** *symmetric with respect to the axis (y'oy).*
  The axis (x'ox) is a ***horizontal tangent****:* Function $x \to x^{2p}$ is even and has an ***extremum***.

- *When* $n \in \mathbb{N}^*$ *is* ***odd*** *the curve is* ***symmetrical with respect to the origin****.*
  The axis (x'ox) is a ***horizontal tangent which crosses the curve****:* Function $x \to x^{2p+1}$ is odd and has an **inflection point** at the center of the symmetry (0,0).

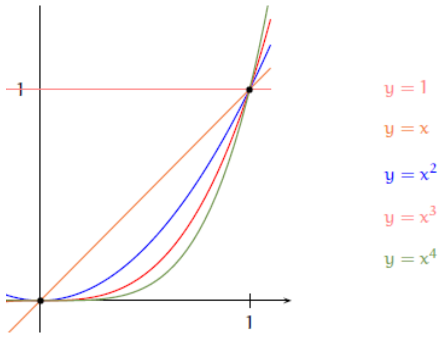
To remember :

- $x \in]0,1[\implies \ 1 > x > x^2 > x^3 > ... \to 0$

i.e. sequence (x^n) is decreasing and tends to 0.

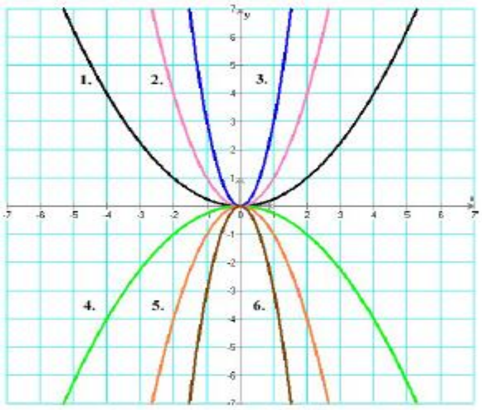
- $x \in]1,+\infty[\implies \ 1 < x < x^2 < x^3 < ... \to +\infty$

i.e. sequence (x^n) is increasing and tends towards $+\infty$.



Exercise

Each of the following six parabolas is the graphical representation of a function of the type $f(x) = ax^2$. Determine, for each of them, the value of the real a.



Correction

To determine a, it suffices to fix a point $M(x, f(x))$of the curve and replace its coordinates in $f(x) = ax^2$.

1)$M(-4,4) \implies a = \frac{1}{4}$, 2)$M(-2,4) \implies a = 1$,

3)$M(1,3) \implies a = 3$, 4)$M(-4,-4) \implies a = -\frac{1}{4}$,

5) inverse of C2: $a = -1$, 6) inverse of C3: $a = -3$.

1.2. Quadratic polynomial function

The **quadratic polynomial function** *is defined on* $\mathbb{R}$ *by*

$$\forall x \in \mathbb{R} \;:\; f(x) = ax^2 + bx + c\;,\; a \neq 0.$$

Canonical form: Let be $a, b, c \in \mathbb{R}$, $a \neq 0$ and put $\Delta = b^2 - 4ac$. $\forall x \in \mathbb{R}$, we have:

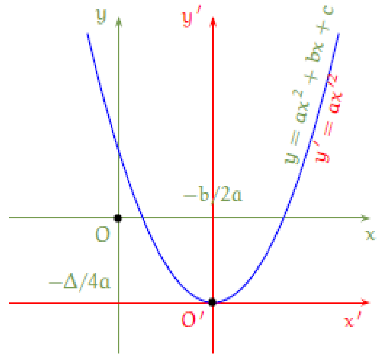
$$\begin{aligned} y = f(x) = ax^2 + bx + c &= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]\;; \end{aligned}$$

We deduce $$y + \frac{\Delta}{4a} = a\,\left(x + \frac{b}{2a}\right)^2.$$

Note that, by ***making a translation*** of the reference $(O, \vec{i}, \vec{j})$ we get a new reference $(O', \vec{i}, \vec{j})$ with $O'(-\frac{b}{2a}, -\frac{\Delta}{4a})$.

This corresponds to the change of variable

$$\begin{cases} x' = x + \dfrac{b}{2a} \\ y' = y + \dfrac{\Delta}{4\,a} \end{cases}$$



then the equation of the curve becomes simpler:

$$y' = a\ x'^2 .$$

So, one easily deduce the curve of f.

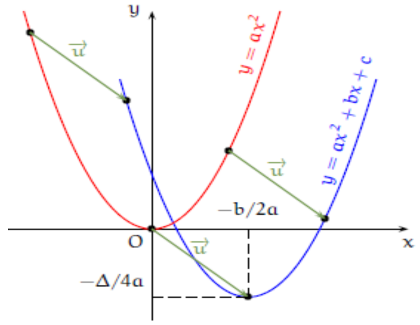
Notice : We can see that the curve of

$$x \to ax^2 + bx + c$$

is the translated of that of known curve of

$$x \to ax^2$$

with a translation vector $\vec{u} = \begin{pmatrix} -b/2a \\ -\Delta/4a \end{pmatrix}$.

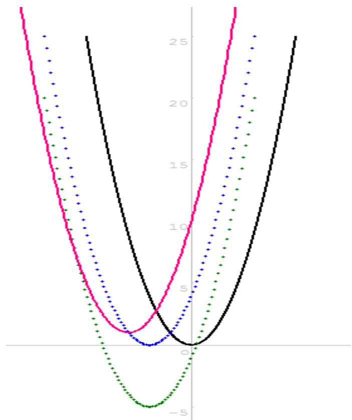


Exercise

Plot in the same Cartesian plane the graph of the functions defined by

$$f(x) = x^2, \quad g(x) = (x+2)^2, \quad h(x) = (x+2)^2 - 5, \qquad k(x) = x^2 + 6x + 10.$$

Correction



$x \to f(x) = x^2$: parabola (Pf).

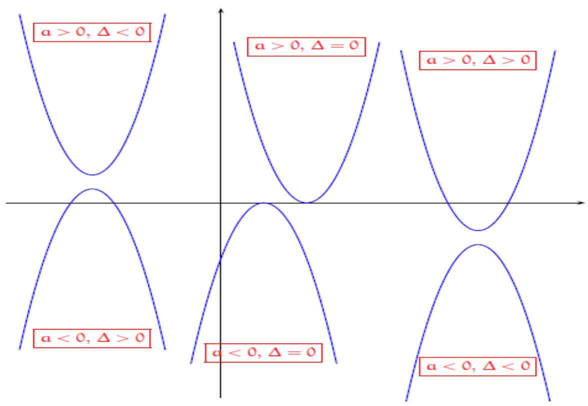
$x \to g(x) = (x+2)^2$: parabola (Pg)= translated 2 left units of (Pf).

$x \to h(x) = (x+2)^2 - 5$: parabola (Ph) = translated 5 units down from (Pg).

$x \to k(x) = x^2 + 6x + 10 = (x^2 + 2 \times 3\,x + 3^2) + 1 = (x+3)^2 + 1$:

parabola (Pk) =translated from vector $\vec{v}\begin{pmatrix} -3 \\ +1 \end{pmatrix}$ of (Pf).

Notice: We can deduct from the curve of $x \to ax^2 + bx + c$ the existence of the roots and the sign of the polynomial $ax^2 + bx + c$. We can also see the intervals of monotony as well as the extremums.



1.3. Sign of the 2nd degree polynomial

We consider the discriminant$\Delta = b^2 - 4ac$.

1. If $\Delta = 0$: $ax^2 + bx + c$ admits a double root $x_1 = x_2 = -\dfrac{b}{2a}$.		
x	$-\infty$ $\qquad -\frac{b}{2a}$	$+\infty$
$ax^2 + bx + c$	(+a) sign	(+a) sign

2. If $\Delta > 0$: $ax^2 + bx + c$ $(a \neq 0)$ has two distinct roots:

$$x' = \frac{-b - \sqrt{\Delta}}{2a} \quad \text{and} \quad x'' = \frac{-b + \sqrt{\Delta}}{2a}.$$

Noting $x_1 = \min(x', x'')$ and $x_2 = \max(x', x'')$ we have

x	$-\infty$ $\qquad x_1$	x_2	$a_n \sim b_n$
$ax^2 + bx + c$	(+a) sign	sign of (-a)	(+a) sign

3. If $\Delta < 0$: $ax^2 + bx + c$ $(a \neq 0)$ does not admit real roots.	
x	$-\infty$ $\qquad\qquad a_n \sim b_n$
$ax^2 + bx + c$	(+a) sign

Notice: when $\Delta \geq 0$ we have $ax^2 + bx + c = a(x - x_1)(x - x_2)$ and

$$x_1 + x_2 = -\frac{b}{a} \quad , \quad x_1 \times x_2 = \frac{c}{a}.$$

Exercise

Find all the solutions in $\mathbb{R}$ of the following inequalities:

1)$x^2 - 4|x| - 5 > 0$ $\qquad$ 2)$|4 - x^2| - |3 - x| > x$.

Correction:

I)

x	$-\infty$ $\qquad\qquad 0$	$+\infty$
$x^2 - 4\|x\| - 5 > 0$	$x^2 + 4x - 5 > 0$	$x^2 - 4x - 5 > 0$
analysis	$\Delta = (4)^2 - 4(1)(-5) = 36$, $x_1 = \frac{-(4)-6}{2(1)} = -5$, $x_2 = \frac{-4+6}{2} = 1$	$\Delta = (-4)^2 - 4(1)(-5) = 36$, $x_1 = \frac{-(-4)-6}{2(1)} = -1$, $x_2 = \frac{4+6}{2} = 5$
Solutions by intervals	$\left(]-\infty, -5[\cup]1, +\infty[\right)\cap]-\infty, 0]$ $S_1 =]-\infty, -5[$	$\left(]-\infty, -1[\cup]5, +\infty[\right)\cap]0, +\infty[$ $S_2 =]5, +\infty[$
Solutions	$S = S_1 \cup S_2 =]-\infty, -5[\cup]5, +\infty[$	

2)Let us first simplify the expression $|4-x^2|-|3-x|>x$:

x	$-\infty$ $\quad -2$	$+2$	3	$+\infty$
$\|4-x^2\|$	$4-x^2$	$-4+x^2$	$4-x^2$	$4-x^2$
$-\|3-x\|$	$3-x$	$3-x$	$3-x$	$-3+x$
inequality	$4-x^2+3-x>x$ i.e., $-x^2-2x+7>0$	$-4+x^2+3-x>x$ i.e., $x^2-2x-1>0$	$4-x^2+3-x>x$ i.e., $-x^2-2x+7>0$	$4-x^2-3+x>x$ i.e., $-x^2+1>0$

We will distinguish the different cases to solve the inequality:

i) If $x\in]-\infty,-2]$ the inequality is $-x^2-2x+7>0$:

$\Delta=(-2)^2-4(-1)(7)=32,\ x_1=1-2\sqrt{2},\ x_2=1+2\sqrt{2}.$

x	$-\infty$ $\quad -2$	$1-2\sqrt{2}$	$+\infty$
$-x^2-2x+7$	-	+	-
Solutions	No solutions		

ii) Whether $x\in]-2,+2]$ the inequality is $x^2-2x-1>0$:

$\Delta=(-2)^2-4(1)(-1)=8,\ x_1=1-\sqrt{2},\ x_2=1+\sqrt{2}.$

x	-2 $\quad 1-\sqrt{2}$	2	$+\infty$
x^2-2x-1	+	-	+
Solutions	$S_1=]-2,1-\sqrt{2}[$	No solutions	

iii) Whether $x\in]2,3]$ the inequality is $-x^2-2x+7>0$:

$\Delta=(-2)^2-4(-1)(7)=32,\ x_1=1-2\sqrt{2},\ x_2=1+2\sqrt{2}.$

x	$1-2\sqrt{2}$ $\quad 2$	3	$1+2\sqrt{2}$
$-x^2-2x+7$	+	+	+
Solutions		$S_2=]2,3[$	

iv)Whether $x\in]3,+\infty]$ the inequality is $-x^2+1>0$. $x_1=1$, $x_2=-1$.

x	-1 $\quad 1$	3	$+\infty$
$-x^2+1$	+	-	-
Solutions			No solutions

We deduce the set of solutions of $|4-x^2|-|3-x|>x$ is

$$S=S_1\cup S_2=]-2,1-\sqrt{2}[\cup]2,3[.$$

2 RATIONAL FUNCTIONS

The **inverse function of power** *is defined on* $\mathbb{R}$ *by*

$$\forall x\in\mathbb{R}^*\ :\ f(x)=\frac{1}{x^n}\ ,\ n\in\mathbb{N}^*.$$

- *It is continuous and indefinitely differentiable on* D_f, *we have:*

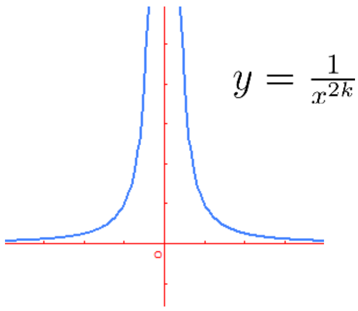
$(\frac{1}{x^n})'=(x^{-n})'=-n\,x^{-n-1}=\frac{-n}{x^{n+1}}$

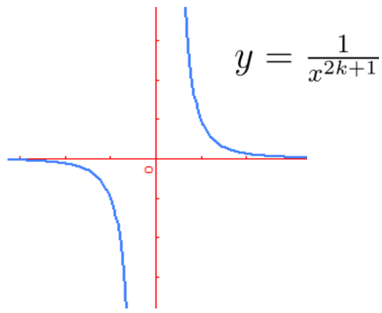
$(\frac{1}{x^n})''=(\frac{-n}{x^{n+1}})'=-n\,(x^{-n-1})'=(-1)^2\,n(n+1)\,x^{-n-2}=\frac{(-1)^2\,n(n+1)}{x^{n+2}}$

$(\frac{1}{x^n})'''=\frac{(-1)^3\,n(n+1)(n+2)}{x^{n+3}}\ \ldots$

- *Its graph is a* **hyperbola** *equilateral having the axes as asymptotes.*
- *The variation of* x^n *depends on the parity of* $n\in\mathbb{N}^*$. *If* $n=2k\ ,\ k\in\mathbb{N}^*$ *then* $x\to x^n$ *decreasing on* $]-\infty,0)$ *and growing on* $(0,+\infty[$.

  We deduce by inverting: $0<a<b\implies\frac{1}{a}>\frac{1}{b}.$
-





Exercise

Find all the solutions in $\mathbb{R}$ of the following inequality:

$$\frac{1}{x+2}-\frac{1}{x-2}<1+\frac{1}{4-x^2}.$$

Correction

Note I the inequality, then I is defined if and only if $x\in\mathbb{R}\setminus\{-2,+2\}$ in this case we have

$$I \iff \frac{x-2}{x^2-4} - \frac{x+2}{x^2-4} < \frac{x^2-4}{x^2-4} - \frac{1}{x^2-4}$$

$$\iff \frac{-4}{x^2-4} < \frac{x^2-5}{x^2-4}$$

$$\iff \frac{x^2-1}{x^2-4} > 0$$

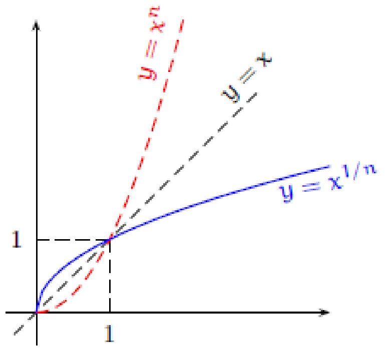
x	$-\infty$ … -2	-2 … -1	-1 … $+1$	$+1$ … $+2$	$+2$ … $+\infty$
x^2-1	$+$	$+$	$-$	$+$	$+$
x^2-4	$+$	$-$	$-$	$-$	$+$
$\frac{x^2-1}{x^2-4}$	$+$	$-$	$+$	$-$	$+$

We deduce the set of solutions of I: $S =]-\infty, -2[\cup]-1, +1[\cup]2, +\infty[$.

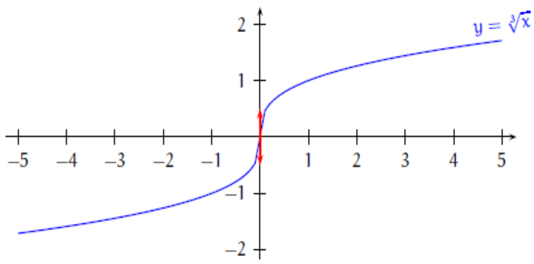
3 IRRATIONAL FUNCTIONS

If $n = 2k$, $k \in \mathbb{N}^*$ *[resp.* $n = 2k+1$, $k \in \mathbb{N}^*$*], function* $x \to x^n$, $n \geq 2$ *is* **bijective** *from* $[0, +\infty[$ *to* $[0, +\infty[$ *[resp. from* $\mathbb{R}$ *to* $\mathbb{R}$*].*

- *Her* **reciprocal** *is the* ***n-th root function*** *denoted* $\sqrt[n]{\ }$
- *In* $[0, +\infty[$ *we have:* $y = \sqrt[n]{x} \iff x = y^n$.
  This remains ***true in*** $\mathbb{R}$ ***when*** $n = 2k+1$, $k \in \mathbb{N}^*$.



Note the tangent to the origin. $x \to \sqrt[3]{x}$ is not differentiable at 0. Why?



Important:

- *If* n *is odd* *then* $\sqrt[n]{x^n} = x$
- *If* n *is even* *then* $\sqrt[n]{x^n} = |x|$

Exercise

1) for real a and b solve $a^2 = b^2$, $a^3 = b^3$, $\sqrt{a} = b$, $\sqrt[3]{a} = b$.

2) for real a and b complete the following implications $a < b \implies a^2 \ ? \ b^2 \ , \ a^3 \ ? \ b^3$.

Correction

1) for a and b real, we have:

- $a^2 = b^2 \iff a^2 - b^2 = (a-b)(a+b) = 0 \iff a = b \vee a = -b,$
- $a^3 = b^3 \iff \sqrt[3]{a^3} = \sqrt[3]{b^3} \iff a = b\,,$
- $\sqrt{a} = b \iff \begin{cases} b \geq 0 \\ a = b^2, \end{cases}$
- $\sqrt[3]{a} = b \iff a = b^3.$

2) for a and b real, we have:

$a < b \implies \begin{cases} a^2 > b^2 & \text{si } a < b \leq 0 \\ a^2 < b^2 & \text{si } 0 \leq a < b \end{cases}$ because the function $x \to x^2$ is decreasing for $x \leq 0$ and increasing for $x \geq 0$.

$a < b \implies \ , \ a^3 < b^3$ because the function $x \to x^3$is increasing on.$]-\infty, +\infty[$.

Exercise

solve in $\mathbb{R}$: 1)$\sqrt{2x+21} = 3x - 1$, 2)$\sqrt[3]{x+7} = x + 1$.

Correction

1)$\sqrt{2x+21} = 3x - 1 \iff \begin{cases} 3x - 1 \geq 0 \\ 2x + 21 = (3x-1)^2 \end{cases} \iff \begin{cases} x \in]\frac{1}{3}, +\infty[\\ 9x^2 - 8x - 20 = 0 \end{cases}$'

$\Delta = 784$,$x_1 = 2$, $x_2 = -\frac{9}{10} \notin]\frac{1}{3}, +\infty[$. So $x = 2$ is the unique solution of the equation.

2)$\sqrt[3]{x+7} = x + 1 \iff x + 7 = (x+1)^3 = x^3 + 3x^2 + 3x + 1 \iff x^3 + 3x^2 + 2x - 6 = 0.$

Note that $x = 1$ is solution of the equation, Euclidean division by $(x-1)$ deal to

$$x^3 + 3x^2 + 2x - 6 = (x-1)(x^2 + 4x + 6) = 0.$$

$\Delta = -8 < 0$, so $x = 1$ is the unique solution of the equation.

Exercise

solve in $\mathbb{R}$: 1)$\sqrt[3]{1+x^3} \leq 1 + x$, 2).$\sqrt{1+x^2} \leq 1 + x$.

Correction

1) The function $t \to t^3$ being increasing we have

$$\sqrt[3]{1+x^3} \leq 1 + x \iff 1 + x^3 \leq (1+x)^3 = 1 + 3x + 3x^2 + x^3 \iff 3x^2 + 3x = 3x(x+1) \geq 0,$$

we deduce $3x^2 + 3x \geq 0 \iff x \in]-\infty, -1] \cup [0, +\infty[$.

2) The function $t \to t^2$ being increasing on $[0, +\infty[$ we obtain

$$0 \leq \sqrt{1+x^2} \leq 1 + x \iff \begin{cases} 1 + x \geq 0 \\ 1 + x^2 \leq (1+x)^2 = 1 + 2x + x^2 \end{cases}$$
$$\iff \begin{cases} x \in]-1, +\infty[\\ 2x \geq 0 \end{cases},$$

hence the set of solutions of the inequality is $S = [0, +\infty[$.

Exercise

Let f be the function defined by $f(x) = \sqrt{x^2 + 6x + 8} - \sqrt{x^2 - 1}$.

Give its definition set D_f and study the sign of $f(x)$ for $x \in D_f$.

Correction

1)$x \in D_f \iff \begin{cases} x^2+6x+8 \geq 0 \\ x^2-1 \geq 0 \end{cases}$.

$x^2+6x+8 : \Delta = 4, x_1 = -4,\ x_2 = -2$ then $x^2+6x+8 \geq 0 \iff x \in S_1 :=]-\infty,-4] \cup [-2,+\infty[$.

On the other hand, $x^2-1 \geq 0 \iff x \in S_2 :=]-\infty,-1] \cup [+1,+\infty[$.

We deduce the definition set of the inequality:

$$D_f = S_1 \cap S_2 =]-\infty,-4] \cup [-2,-1] \cup [1,+\infty[.$$

For $x \in D_f$, multiplying and dividing by the conjugate we get

$$f(x) = \frac{(x^2+6x+8)-(x^2-1)}{\sqrt{x^2+6x+8}+\sqrt{x^2-1}} = \frac{6x+9}{\sqrt{x^2+6x+8}+\sqrt{x^2-1}}.$$

The denominator being positive, the sign of $f(x)$ is the same as $6x+9 \mathrm{In} D_f$. We deduce:

$$f(x) = 0 \iff x = -\frac{3}{2};$$

$$f(x) < 0 \iff x \in]-\infty,-4] \cup [-2,-\frac{3}{2}[;$$

$$f(x) = 0 \iff x \in]-\frac{3}{2},-1] \cup [1,+\infty[.$$

Exercise

solve in $\mathbb{R}$: 1)$\sqrt{x+1}-\sqrt{x-2} \geq 3$, 2)$\dfrac{\sqrt{1-9x^2}+2x}{3x-2} > 0$, 3)$\sqrt{\dfrac{x^2+8|x|-9}{x^2-1}} > x-3$.

Directions

First find the domain of definition. 1) Leave a single radical to the left of the inequality then square to get rid of a radical. For the second you can do the same or multiply and divide by the conjugate expression then study the sign.

2) Study separately the signs of the numerator and the denominator; summarize in a table and deduce the sign of the fraction.

3) Get rid of the radical by an equivalence; we will obtain a system of inequalities to solve separately then consider the intersection. Pay attention to the absolute value which requires a separation of cases.

3.1. Rational exponent power

Definition:

We can extend the power function to $n = \frac{p}{q}$, $p \in \mathbb{Z}, q \in \mathbb{N}^*$*:*

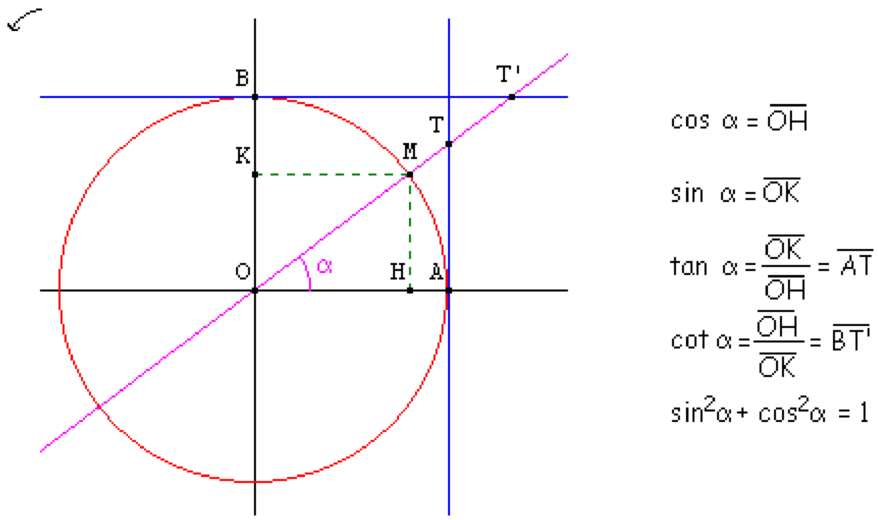
$$x^{\frac{p}{q}} := \sqrt[q]{x^p} \qquad \textit{and} \quad x^0 := 1.$$

Especially $\sqrt{x} = x^{\frac{1}{2}}$ *and* $\forall n \geq 2$: $\sqrt[n]{x} = x^{\frac{1}{n}}$.

4 CIRCULAR FUNCTIONS

To correctly handle circular sine functions, cosine tangent... it is imperative to know ***how to work with the trigonometric circle*** (to simplify, it will be noted C_T). ***It is the circle cantered at the origin of the reference (orthonormal) and of radius the unit of the reference***.

Consider the point $A(1,0)$ which will be ***origin for the arcs*** of C_T. For$\alpha \in \mathbb{R}$ we associate a point $M(\alpha) \in C_T$ such as $mes(\widehat{AOM}) = \alpha$.



Cercle trigonométrique : OM=1

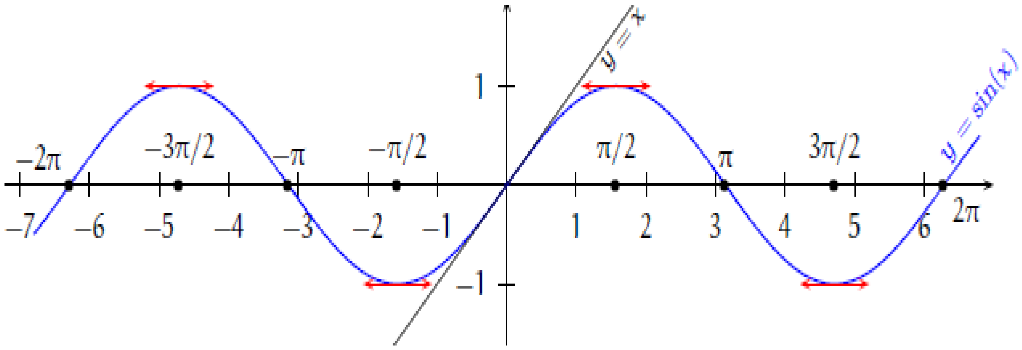
From the trigonometric circle we can see the following for any $x \in \mathbb{R}$

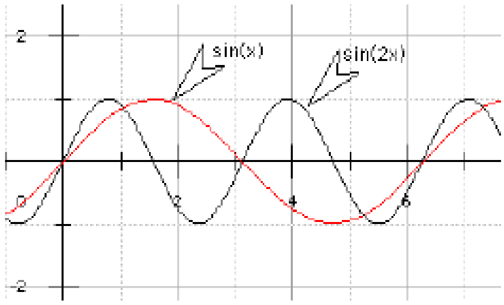
4.1. Sine function:

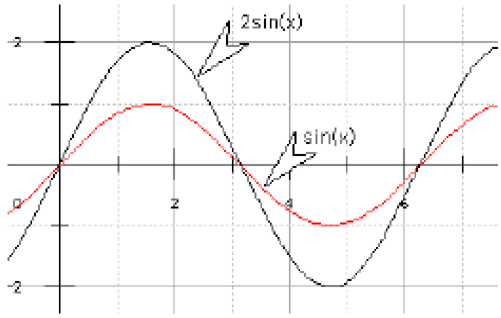
Definition and properties:

- $\sin x$ *is the **ordinate** of the point* $M(x) \in C_T$;
- $\sin x$ *is increasing on*$(-\frac{\pi}{2}, +\frac{\pi}{2})$;
- $2\pi-$***periodical*** *i.e.* $\forall x \in \mathbb{R}\ :\ \sin(x + k\,2\pi) = \sin x\ ,\ k \in \mathbb{Z}$;
- $\sin x$ *is **bounded**:* $\forall x \in \mathbb{R}\ :\ -1 \le \sin x \le +1.$
- $\sin 0 = \sin \pi = 0,\quad \cos\frac{\pi}{2} = 1,\quad \cos(-\frac{\pi}{2}) = -1.$
- $x \to \sin x$ *is continuous and differentiable on*$\mathbb{R}$:

$$(\sin x)' = \cos x.$$





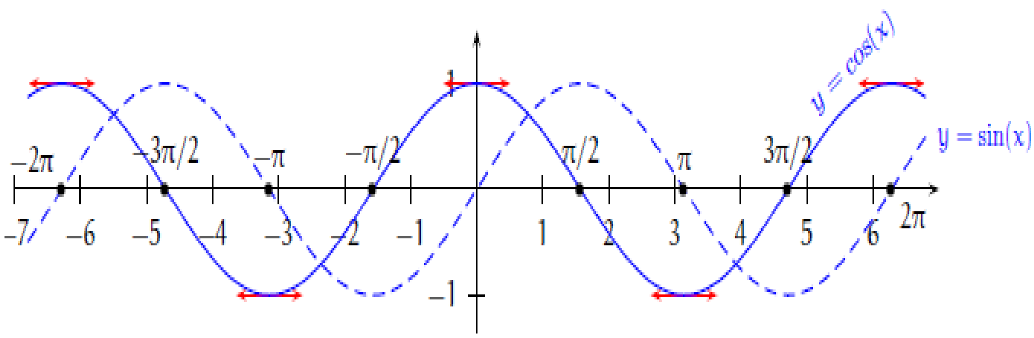


4.2. Cosine function:

Definition and properties:

- $\cos x$ *is the* ***abscissa*** *of the point* $M(x) \in C_T$;
- $\cos x$ *is decreasing on* $(0, \pi)$;
- $2\pi-$***periodical*** *i.e.* $\forall x \in \mathbb{R} \;:\; \cos(x + k\,2\pi) = \cos x \;,\; k \in \mathbb{Z}$;
- $\cos x$ *is* ***bounded*** $\forall x \in \mathbb{R} \;:\; -1 \leq \cos x \leq +1$.
- $\cos 0 = 1$, $\cos \pi = -1$, $\cos \frac{\pi}{2} = \cos(-\frac{\pi}{2}) = 0$.
- $x \to \cos x$ *is continuous and differentiable on* $\mathbb{R}$:

$$(\cos x)' = -\sin x.$$



- *The Pythagorean theorem shows that*

$$\forall x \in \mathbb{R} \;:\; \cos^2 x + \sin^2 x = 1.$$

Exercise

The evolution of the population P of deer is modelled by the function:

$$P(t) = 4000 + 500\ \sin(2\pi t - \pi/2)$$

where t is measured in years.

a. What is the period of the function P(t)?

b. When in the year is the population at its peak? What is the population at that time?

vs. Is there a minimum? If yes, when?

Correction

a. Let T denote the period:

$$\begin{aligned} P(t+T) = P(t)\ ,\ \forall t \in \mathbb{R} &\iff \sin(2\pi t + 2\pi T - \pi/2) = \sin(2\pi t - \pi/2) \\ &\iff 2\pi t + 2\pi T - \pi/2 = 2\pi t - \pi/2 + 2\pi \\ &\iff T = 1 \end{aligned}$$

,

b. $P'(t) = 4000 + 500\ \sin(2\pi t - \pi/2) = 500\ \cos(2\pi t - \pi/2)$

$$P'(t) = 0 \iff 2\pi t - \pi/2 = \pi/2 + 2k\pi \quad \vee \quad 2\pi t - \pi/2 = -\pi/2 + 2k\pi \iff t = 1/2 + k \quad \vee \quad t = 1 + k.$$

0		1/2		1		3/2
$P'(t)$	$+$		$-$		$+$	

the population peaks in the middle of each year: $P(1/2) = 4000 + 500\ \sin(\pi/2) = 4500.$

Note: a minimum is reached at the beginning of each year, i.e.,
$P(1) = 4000 + 500\ \sin(3\pi/2) = 3500.$

4.3. Tangent, cotangent function:

Definition and properties(tangent):

- $\tan x := \dfrac{\sin x}{\cos x}\ ,\quad x \neq \frac{(2k+1)\pi}{2}\ ,\ k \in \mathbb{Z}$
- It is is continuous and differentiable on D_f

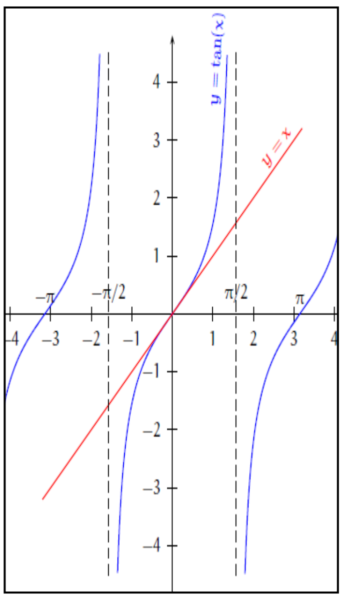
$$\big(\tan x\big)' = \frac{1}{1+x^2}.$$

- $\tan x$ is increasing on $(-\frac{\pi}{2}, +\frac{\pi}{2})$;
- π**−periodical** i.e.

$$\forall x \in D_f\ \ :\ tan(x + k\,\pi) = tan(x)\ ,\ k \in \mathbb{Z};$$

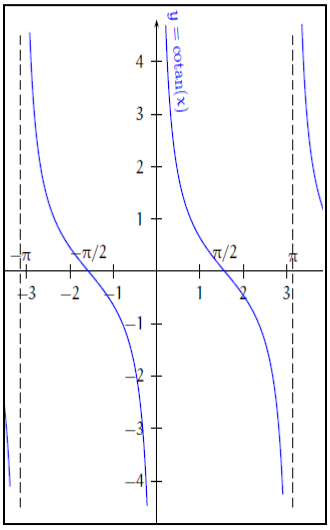
- $\tan x$ is **not bounded**: $\lim\limits_{x \overset{>}{\to} -\frac{\pi}{2}} \tan x = -\infty$

and $\lim\limits_{x \overset{<}{\to} +\frac{\pi}{2}} \tan x = +\infty$.



Definition (cotangent):

$$cotan\, x := \frac{1}{\tan x} = \frac{\cos x}{\sin x}\ x \neq \frac{k\pi}{2}\ ,\ k \in \mathbb{Z}$$



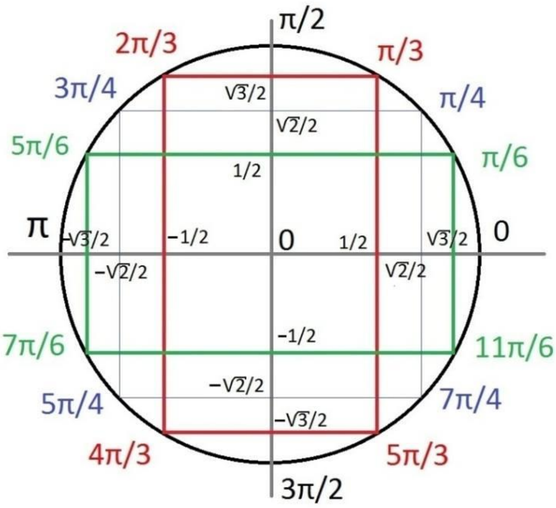
Comparisons:

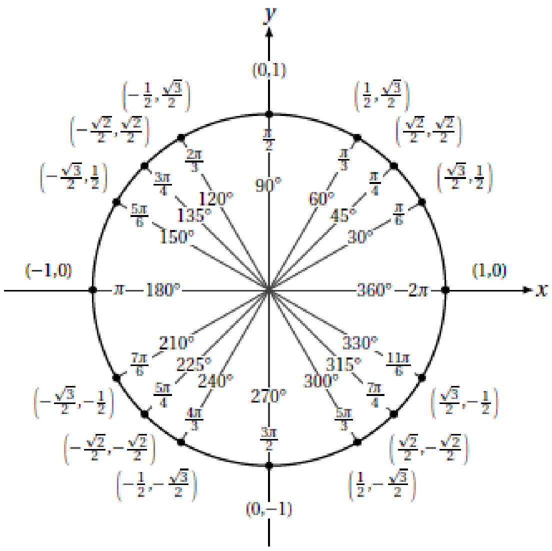
$\forall x \in]0, \frac{\pi}{2}[:\ \sin x \leq x \leq \tan x$ *and then.* $\forall x \in]0, \frac{\pi}{2}[:\ \cos x \leq \dfrac{\sin x}{x} \leq 1.$

We deduce $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ *i.e.* $\sin x \sim x$ *close to 0.*

4.4. Trigonometric values of particular arcs

One must know the sines and cosines for certain particular arcs (or angles) (expressed in radians) and know how to deduce from the trigonometric circle the variations and signs of the basic circular functions.





Exercise

solve in $\mathbb{R}$:

1) $\sin x > \frac{\sqrt{3}}{2}$, 2) $2\cos^2(x) + \cos(x) > 1$, 3) $\dfrac{1 - 2\sin(x)}{1 - 2\cos(x)} \leq 0$, 4) $\dfrac{\sin(x)}{\sqrt{2\sin(x) - 1}} \geq 1$.

Correction

1) The function $x \rightarrow \sin x$ is even, we restrict the study to $(0, 2\pi)$.

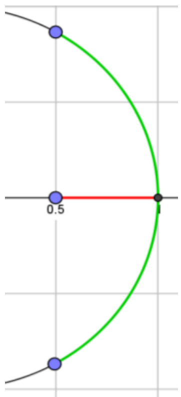
$$\sin x = \frac{\sqrt{3}}{2} \iff x = \frac{\pi}{3} \vee x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

From the trigonometric circle we see that:

$\sin x > \frac{\sqrt{3}}{2} \iff \frac{\pi}{3} < x < \frac{2\pi}{3}.$

In$\mathbb{R}$: $\quad \sin x > \frac{\sqrt{3}}{2} \iff \frac{\pi}{3} + 2k\pi < x < \frac{2\pi}{3} + 2k\pi \ , \ k \in \mathbb{Z}.$

2)) The function $x \rightarrow \sin x$ is even, we restrict the study to $(-\pi, +\pi)$.



We pose $X = \cos(x)$, $\quad 2\cos^2(x) + \cos(x) > 1$ will be

$2X^2 + X - 1 > 0$. $\Delta = 9, X_1 = -1, X_2 = \frac{1}{2}$; then

$2X^2 + X - 1 > 0 \iff X < -1 \ (impossible) \ \vee X > \frac{1}{2}.$

$$\cos(x) = \frac{1}{2} \iff x = \frac{\pi}{3} \vee x = -\frac{\pi}{3}.$$

From the trigonometric circle we see that

$$\cos x > \frac{1}{2} \iff -\frac{\pi}{3} < x < \frac{\pi}{3}.$$

In $\mathbb{R}$: $\quad \cos x > \frac{1}{2} \iff -\frac{\pi}{3} + 2k\pi < x < \frac{\pi}{3} + 2k\pi \ , \ k \in \mathbb{Z}.$

3) We restrict the study to $(-\pi, +\pi)$. Let's study the sign of $\dfrac{1 - 2\sin(x)}{1 - 2\cos(x)}$.

$$1 - 2\sin(x) = 0 \iff \sin(x) = \frac{1}{2} \iff x = \frac{\pi}{6} \vee x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

and $\quad 1 - 2\sin(x) < 0 \iff \sin(x) > \frac{1}{2} \iff \frac{\pi}{6} < x < \frac{5\pi}{6}.$

$$1 - 2\cos(x) = 0 \iff \cos x = \frac{1}{2} \iff x = \frac{\pi}{3} \vee x = -\frac{\pi}{3}$$

and $\quad 1 - 2\cos(x) < 0 \iff \cos(x) > \frac{1}{2} \iff -\frac{\pi}{3} < x < \frac{\pi}{3}.$

From the trigonometric circle we see that:

x	$-\pi$ $\quad$ $-\frac{\pi}{3}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{5\pi}{6}$	$+\pi$
$1 - 2\sin(x)$	$+$	$+$	$-$	$-$	$+$
$1 - 2\cos(x)$	$+$	$-$	$-$	$+$	$+$
$\dfrac{1 - 2\sin(x)}{1 - 2\cos(x)}$	$+$	$-$	$+$	$-$	$+$

We deduce $\quad \dfrac{1 - 2\sin(x)}{1 - 2\cos(x)} \leq 0 \iff x \in \left] -\dfrac{\pi}{3}, \dfrac{\pi}{6} \right] \cup \left] \dfrac{\pi}{3}, \dfrac{5\pi}{6} \right].$

In $\mathbb{R}$: $\quad \dfrac{1 - 2\sin(x)}{1 - 2\cos(x)} \leq 0 \iff x \in \left] -\dfrac{\pi}{3} + 2k\pi, \dfrac{\pi}{6} + 2k\pi \right] \cup \left] \dfrac{\pi}{3} + 2k\pi, \dfrac{5\pi}{6} + 2k\pi \right] \ , \ k \in \mathbb{Z}.$

4) We restrict the study to$(-\pi, +\pi)$. Let us find the domain of definition D.

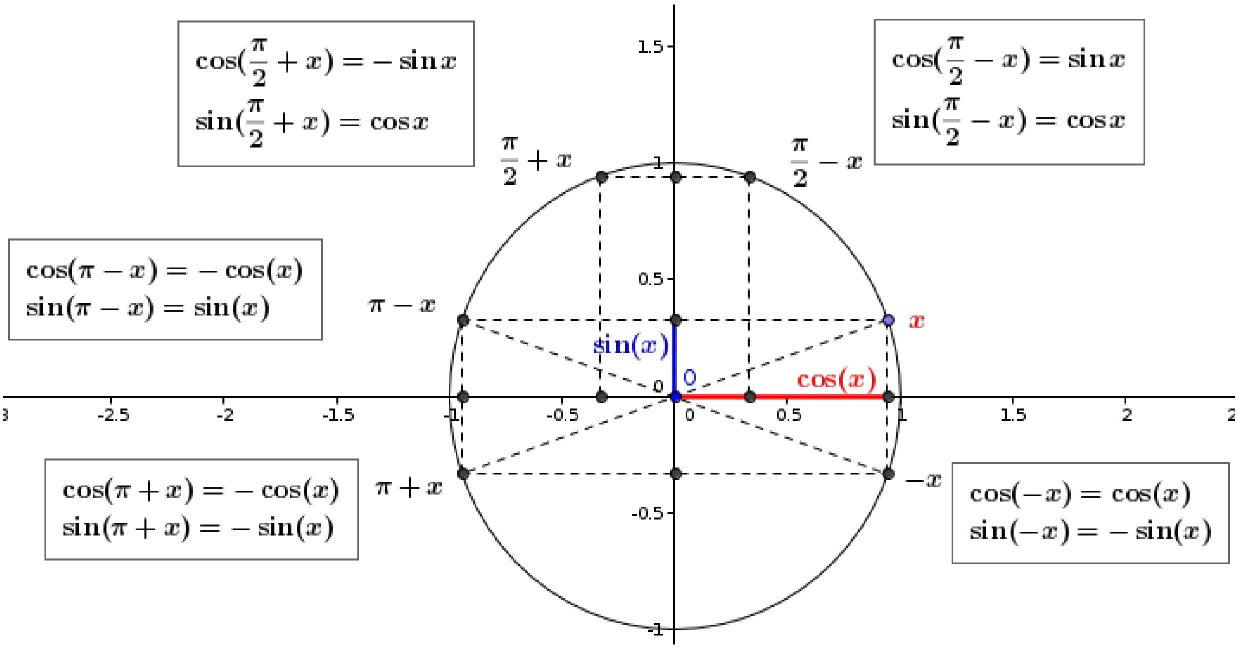
$x \in D \iff 2\sin(x) - 1 > 0 \iff \sin(x) > \frac{1}{2} \iff \frac{\pi}{6} < x < \frac{5\pi}{6}$; hence for $x \in D$ we have

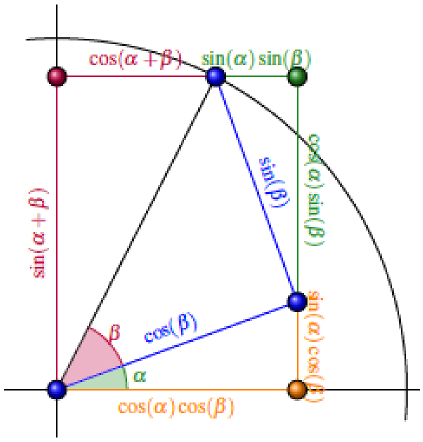
$$\frac{\sin(x)}{\sqrt{2\sin(x) - 1}} \geq 1 \iff \sin(x) - \sqrt{2\sin(x) - 1} = \frac{\sin^2(x) - 2\sin(x) + 1}{\sin(x) + \sqrt{2\sin(x) - 1}} = \frac{(\sin(x) - 1)^2}{\sin(x) + \sqrt{2\sin(x) - 1}} \geq 0.$$

The set of solutions is $S|_p = \left] \dfrac{\pi}{6}, \dfrac{5\pi}{6} \right[.$ $\qquad$ In $\mathbb{R}$ the set of solutions is $S = \left] \dfrac{\pi}{6} + 2k\pi, \dfrac{5\pi}{6} + 2k\pi \right[\ , \ k \in \mathbb{Z}.$

4.5. Trigonometric relations:

Note 1: We must know the formulas of basic transformations.





(1) Basic things to remember

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$$

(2) By replacing in (1) b by $-b$ we obtain

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$
$$\tan(a-b) = \frac{\tan(a)-\tan(b)}{1-\tan(a)\tan(b)}$$

(3) By replacing in (1) b by a we obtain

$$\cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a)$$
$$\sin(2a) = 2\sin(a)\cos(a)$$
$$\tan(2a) = \frac{2\tan(a)}{1-\tan^2(a)}$$

(4) Summing (1) and (2) then putting $a+b \mapsto a$ ***And*** $a-b \mapsto b$ ***we obtain***

$$\cos(a)+\cos(b) = 2\cos\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right) \qquad \cos(a)-\cos(b) = -2\sin\left(\frac{a-b}{2}\sin\frac{a+b}{2}\right)$$

$$\sin(a)+\sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \qquad \sin(a)-\sin(b) = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

(5) From (4) we deduce

$$\cos(a)\cos(b) = \frac{\cos(a+b)+\cos(a-b)}{2}$$
$$\sin(a)\sin(b) = \frac{\cos(a-b)-\cos(a+b)}{2}$$
$$\sin(a)\cos(b) = \frac{\sin(a+b)+\sin(a-b)}{2}$$

$$\cos^2(a) = \frac{1+\cos(2a)}{2} \qquad \sin^2(a) = \frac{1-\cos(2a)}{2}$$

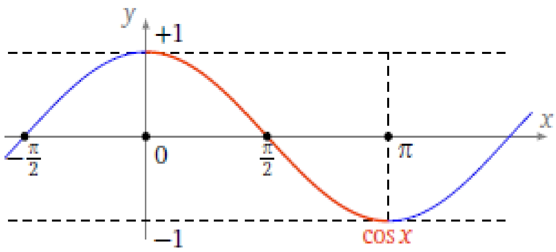
By setting we get $t := \tan\left(\frac{a}{2}\right)$, alors

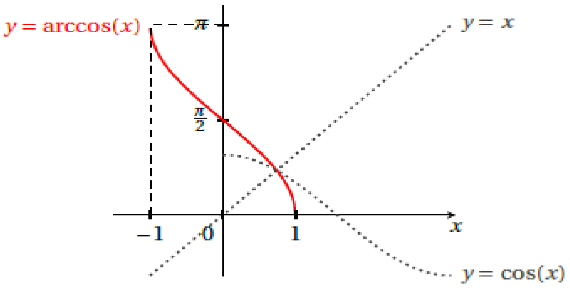
$$\cos(a) = \frac{1-t^2}{1+t^2} \qquad \sin(a) = \frac{2t}{1+t^2} \qquad \tan(a) = \frac{2t}{1-t^2}$$

5 INVERSE CIRCULAR FUNCTIONS

5.1. arc cosine

The restriction $\cos : [0,\pi] \to [-1,+1]$ *is a bijection*	*Its inverse bijection is the function* ***arccosine.***





Arc cosine function:

$$\begin{cases} y = \arccos(x) \\ -1 \leq x \leq +1 \end{cases} \iff \begin{cases} x = \cos(y) \\ 0 \leq y \leq +\pi \end{cases}$$

- *Itis continuous on* $[-1,+1]$ *and differentiable on* $]-1,+1[$, *we have*

$$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}.$$

Proof: We have $\cos(\arccos x) = x$, by differentiating we'll have

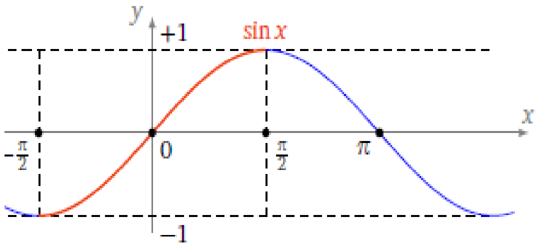
$$-\arccos' x \times \sin(\arccos x) = 1$$

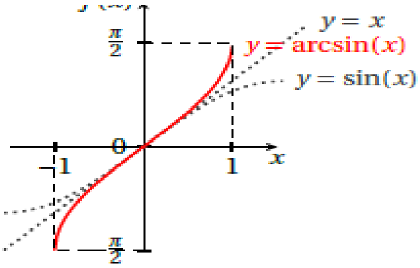
then, $$\arccos' x = \frac{-1}{\sin(\arccos x)} = \frac{-1}{\sqrt{1-\cos^2(\arccos x)}} = \frac{-1}{\sqrt{1-x^2}}.$$

5.2. arc sine

The restriction $\sin : [-\frac{\pi}{2}, +\frac{\pi}{2}] \to [-1,+1]$ *is a bijection*

Its inverse bijection is the function ***arsinus***.





Arc sine function:

$$\begin{cases} y = \arcsin(x) \\ -1 \le x \le +1 \end{cases} \iff \begin{cases} x = \sin(y) \\ -\frac{\pi}{2} \le y \le +\frac{\pi}{2} \end{cases}$$

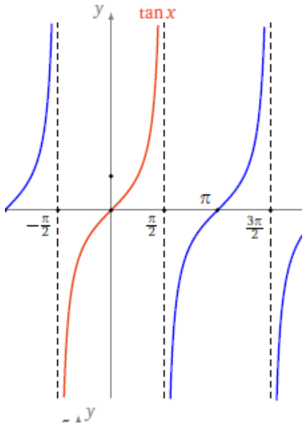
- *It is continuous on* $[-1,+1]$ *and differentiable on* $]-1,+1[$, *we have*

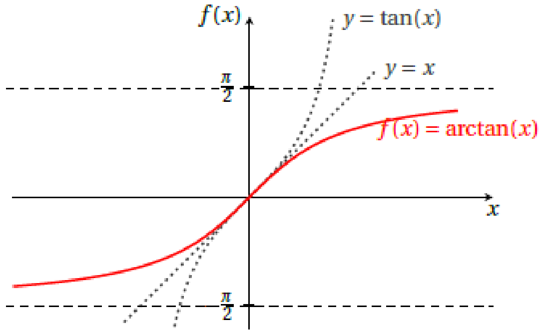
$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}.$$

5.3. arc tangent

The restriction $\tan :]-\frac{\pi}{2}, +\frac{\pi}{2}[\to \mathbb{R}$ *is a bijection*

Its inverse bijection is the function ***arctangent***.





Tangent arc function:

$$\begin{cases} y = \arctan(x) \\ x \in \mathbb{R} \end{cases} \iff \begin{cases} x = \tan(y) \\ -\frac{\pi}{2} < y < +\frac{\pi}{2} \end{cases}$$

- *It is continuous and differentiable on* $\mathbb{R}$*, we have*

$$(\arctan x)' = \frac{1}{\sqrt{1+x^2}}.$$

Remember:

$$\forall x \in [-1,1] \quad \arcsin(x) + \arccos(x) = \frac{\pi}{2}$$

$$\sin(\arccos(x)) = \sqrt{1-x^2} = \cos(\arcsin(x))$$

$$\forall x > 0 \quad \arctan(x) + \arctan(1/x) = \frac{\pi}{2}$$

$$\forall x < 0 \quad \arctan(x) + \arctan(1/x) = -\frac{\pi}{2}$$

6 LOGARITHM AND EXPONENTIAL

6.1. Logarithm

Definition and properties:

- *There is a unique function, denoted* $\ln :]0,+\infty[\to \mathbb{R}$ *such as:*

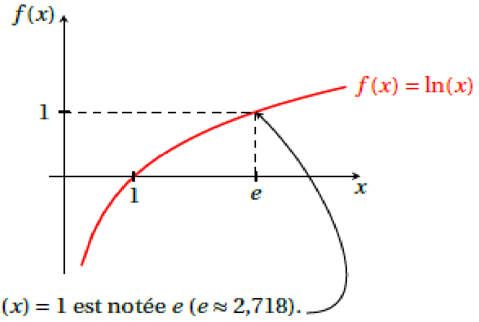
$$\forall x > 0 \ : \ \ln'(x) = \frac{1}{x} \qquad \textit{and} \qquad \ln(1) = 0.$$

It is the primitive of $x \to \frac{1}{x}$ *which vanishes at point 1:*

$$\forall x > 0 \ : \ \ln(x) = \int_1^x \frac{1}{t}\, dt.$$

- $\ln$ *is a **continuous**, strictly **increasing** function and defines a **bijection** between* $]0,+\infty[$ *and* $\mathbb{R}$*.*

- $\ln$ *is a **concave** function and* $\ln(x) \leq x - 1$. *(the line defined by* $y = x - 1$ *is the tangent to* $C_{\ln}$ *at the abscissa point 1)*



L'unique solution de l'équation ln(x) = 1 est notée e (e ≈ 2,718).

Algebraic properties:

This function verifies (for all $a, b > 0$*):*

1. $\ln(a \times b) = \ln a + \ln b$, 2. $\ln(a^n) = n \ln a$, $n \in \mathbb{N}$,

3. $\ln(\frac{a}{b}) = \ln a - \ln b$, *especially* $\ln(\frac{1}{b}) = \ln(b^{-1}) = -\ln b$

Notice: according to 3. we can extend property 2. to the exponents of $\mathbb{Z}$:

$$\ln(a^n) = n \ln a \ , \quad n \in \mathbb{Z}$$

Exercise

Simplify the following expressions:

$$A = \frac{\ln(2x)}{\ln(x)} \quad x \neq 1 \ , \quad B = (\ln(x))^2 - \ln(x^2) \ , \quad C = (\ln(x))^2 - \ln(x^2) + 1.$$

Correction

$A = \frac{\ln(2x)}{\ln(x)} = \frac{\ln(2) + \ln(x)}{\ln(x)} = \frac{\ln(2)}{\ln(x)} + 1.$

$B = (\ln(x))^2 - \ln(x^2) = (\ln(x))^2 - 2\ln(x) = \ln(x)\big(\ln(x) - 2\big).$

$C = (\ln(x))^2 - ln(x^2) + 1 = (\ln(x))^2 - 2\ln(x) + 1 = \big(\ln(x) - 1\big)^2.$

Exercise

solve in $\mathbb{R}$:

1) $\ln(x) - \frac{2}{\ln(x)} \geq 1$, **2)** $\frac{1}{2}\ln(x) + \frac{1}{2}\ln(1 - x) < \ln(2) + \frac{1}{2}\ln(5).$

Correction

1) $\ln(x) - \frac{2}{\ln(x)} \geq 1$, the equation is defined if and only if $x > 0$ and $x \neq 1$; in this case we have

$$\ln(x) - \frac{2}{\ln(x)} \geq 1 \iff \ln^2(x) - \ln(x) - 2 \geq 0.$$

We pose $X = \ln(x)$, we'll get $X^2 - X - 2$, $\Delta = 9$, $X_1 = -1$, $X_2 = 2$ then

$$X^2 - X - 2 \geq 0 \iff X \leq -1 \vee X \geq 2.$$

We deduce $\ln(x) - \dfrac{2}{\ln(x)} \geq 1 \iff \ln(x) \leq -1 \vee \ln(x) \geq 2 \iff 0 < x \leq e^{-1} = \dfrac{1}{e} \vee x \geq e^2$ (we have $x \neq 1$).

2) $\frac{1}{2}\ln(x) + \frac{1}{2}\ln(1-x) < \ln(2) + \frac{1}{2}\ln(5)$, the equation is defined if and only if $\begin{cases} x > 0 \\ 1 - x > 0 \end{cases}$ i.e., $0 < x < 1$;

in this case we'll have

$$\begin{aligned}\frac{1}{2}\ln(x) + \frac{1}{2}\ln(1-x) < \ln(2) + \frac{1}{2}\ln(5) &\iff \ln(x) + \ln(1-x) < 2\ln(2) + \ln(5)\\ &\iff \ln x(1-x) < \ln(20)\\ &\iff x(1-x) < 20\\ &\iff -x^2 + x - 20 < 0\ .\end{aligned}$$

$\Delta = 81$, $x_1 = -4$, $x_2 = +5$ then $\quad -x^2 + x - 20 < 0 \iff x \in \Big(]-\infty, -4[\cup] + 5, +\infty[\Big) \cap]0,1[= \phi$;

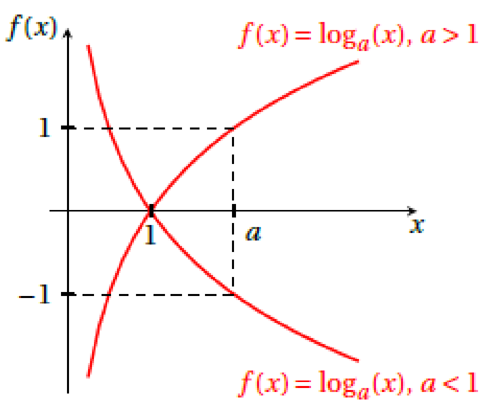
so the inequality has no solution.

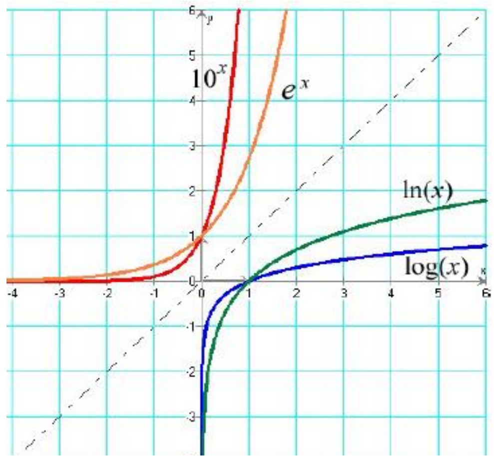
6.2. Logarithm of base a>0

$\ln(x)$ is called the natural logarithm. It is characterized by $\ln(e) = 1$.
We define the logarithm of base a≠1 by

$$log_a(x) := \frac{\ln x}{\ln a}$$

so that $log_a(a) = 1$. For a=10 we'll get the decimal logarithm
$log(.) := \ln_{10}(.)$.





Examples:

1) pH. measures the acidity of a solution. The pH of a solution is defined by $pH = -log_{10}([H^+])$ ($[H^+]$denotes the molar concentration of ions H^+ of the solution. The lower the pH of a solution, the higher its concentration of ions is and more acidic is the solution).

2) In computing, the logarithm in base 2 also intervenes: $log_2(.)$.

Exercise

In acoustics, the intensity of a sound is measured in decibels $I = 10\,log\left(\frac{J}{J_0}\right)$ where J is the acoustic power of the sound (inW/m) and J_0 is the lowest power audible to a human at a frequency of 1 kHz ($J_0 = 12^{-12}W/m$).

The range of intensity perceptible to the human ear goes from 0 dB (the lowest audible power by a human being) to 120 dB which corresponds to the pain threshold. Here is the intensity of some sounds:

Sound of tree leaves	10dB	Ordinary conversation	65dB
Whisper	20dB	Jackhammer at 3m	90dB
Car	50dB	***limit of pain***	***120dB***

Knowing that a loudspeaker with a power of Q watts placed at a distance of R meters from an observer develops an acoustic power of$J = \frac{Q}{4\pi R^2}\ W/m$.

1) Calculate the intensity of the sound.
2) If the loudspeaker power is $100\ W$ calculate the intensity of the sound perceived by an observer located at a distance of 1m. What do you notice?
3) what about for a distance of 10m.

Correction

1) $I = 10\,log\left(\frac{J}{J_0}\right) = 10\,log\left(\frac{Q}{J_0\,4\pi R^2}\right) = 10\left(log(Q) - 2log(R) - log(4\pi) - log(J_0)\right)$

2) If$Q = 100\ W$ And$R = 1\ m$SO$I = 10\left(log(10^2) - 2log(1) - log(4\pi) - log(12^{-12})\right) \approx 129.$

The pain threshold of the human ear is exceeded.

3) If $Q = 100\ W$ and $R = 10\ m$ then $I \approx 129 - 20\,log(10) = 109$ décibels which already exceeds the intensity of the jackhammer at 3m (see table above).

Exercise

solve in $\mathbb{R}$:

1)$log_2\frac{x+\sqrt{x^2+9}}{2x} > 1$ 2)$log_5(x-7) > 2$ 3)$log_{2/3}(x^2-1) > 2.$

Correction

1)$log_2\frac{x+\sqrt{x^2+9}}{2x} > 1$: the inequality is defined for $x \neq 0$.

$$log_2\frac{x+\sqrt{x^2+9}}{2x} > 1 \iff \frac{x+\sqrt{x^2+9}}{2x} > 2^1 \iff \frac{-3x+\sqrt{x^2+9}}{2x} > 0$$

a) If $x < 0$ then $-3x + \sqrt{x^2+9} > 0$ and $2x < 0$, so the problem has no solutions in this case.

b) If$x > 0$ then $2x > 0$ and $-3x + \sqrt{x^2+9} = \dfrac{x^2+9-9x^2}{3x+\sqrt{x^2+9}} = \dfrac{9-8x^2}{3x+\sqrt{x^2+9}}$,

the solution set of the problem is $]0, \frac{3}{2\sqrt{2}}[$.

2)$log_5(x-7) > 2$: the inequality is defined for $x-7>0, D =]7, +\infty[$.

$log_5(x-7) > 2 \iff x-7 > 5^2 \iff x > 32.$

3)$log_{2/3}(x^2-1) > 2$: the inequality is defined for $x^2-1>0$, $D =]-\infty, -1[\cup]+1, +\infty[$.

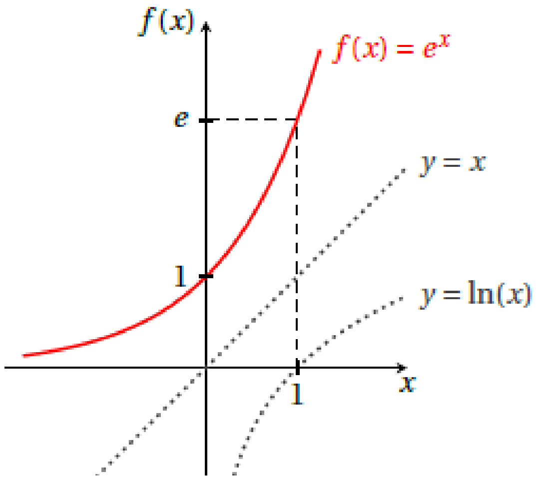
$log_{2/3}(x^2-1) > 2 \iff x^2-1 < \left(\frac{2}{3}\right)^2 \iff x^2 < \frac{13}{9}$, the solution set of the problem is

$$D =]-\sqrt{\tfrac{13}{9}}, -1[\cup]+1, +\sqrt{\tfrac{13}{9}}[.$$

6.3. Exponential

Definition and properties:

- *The reciprocal bijection of* $\ln :]0,+\infty[\to \mathbb{R}$ *is called the denoted* ***exponential function*** $\exp$.
- $\exp$ *is a* ***continuous*** *and* ***indefinitely differentiable*** *function*
  $\exp'(x) = \exp(x)$ *And* $\forall n \geq \mathbb{N}^* \ : \ \exp^{(n)}(x) = \exp(x)$.
- $\exp$ *is a continuous, strictly* ***increasing*** *function and defines a* ***bijection*** *between* $\mathbb{R}$ *and* $]0,+\infty[$.
- $\exp$ *is a* ***convex*** *function and* $\exp(x) \geq x+1$. *(the line defined by $y = x+1$ is the tangent to $C_{\ln}$ at the abscissa point 1)*



Notation: ($\exp(x)$ and e^x)

If $x \in \mathbb{Z}$ then: $\exp(x) = \exp(x \ln e) = \exp(\ln e^x) = e^x$.

We will denote for all $x \in \mathbb{R}$: $\exp(x) = e^x$.

Algebraic properties:

For every $a, b > 0$:

1. $\exp(a+b) = \exp a \times \exp b$, *i.e.* $e^{a+b} = e^a \times e^b$
2. $(\exp a)^n = \exp(n\, a)$, $n \in \mathbb{Z}$, *i.e.* $(e^a)^n = e^{n\, b}$
3. $e^{a-b} = \dfrac{e^a}{e^b}$ *especially* $e^{-b} = \dfrac{1}{e^b}$.

Exercise

The number of bacteria $N(t)$ contained in a culture at time t (expressed in days) is given by $N(t) = N_0\, e^{\beta\, t}$ where N_0 is the initial number of bacteria and βa coefficient depending on the type of bacteria and the surrounding environment.

The number of bacteria in a culture was estimated at 200,000 after 3 days and 1,600,000 after 4.5 days.

a. What is the bacteria count after 5 days?

b. When does the culture contain 800,000 bacteria?

Correction

a. $N(3) = N_0\, e^{3\beta} = 200\,000$ and $N(4.5) = N_0\, e^{4.5\beta} = 1\,600\,000$, we deduce

$\dfrac{N(4.5)}{N(3)} = \dfrac{e^{4.5\beta}}{e^{3\beta}} = e^{1.5\beta} = 8$ then $\beta = \dfrac{\ln(8)}{1.5} \approx 1.386$; Hence $N_0 = \dfrac{200\,000}{e^{3\beta}} \approx 16\,000.$

$N_0 = \dfrac{200\,000}{e^{1.386\times 3}} \approx 3127.76$ We get $N(t) = 3127.76\; e^{1.386\times t}$.

The number of bacteria after 5 days will be $N(5) = 3127.76\; e^{1.386\times 5} \approx 3\,198\,115.$

b. $N(t) = 3127.76\; e^{1.386\, t} = 800\,000 \iff e^{1.386\, t} = \dfrac{800\,000}{3127.76} = 255.774,$

we deduce $t = \dfrac{\ln 255.774}{1.386} \approx 4\ jours$

Exercise

Any radioactive body disintegrates over time. The number of radioactive atoms$N(t)$at time t (in years) is given by$N(t) = N_0\, e^{\mu\, t}$ where N_0 is the number of radioactive atoms at time $t = 0$ and μ a coefficient depending on the material.

All living beings contain a constant proportion of carbon 14 atoms (a radioactive isotope of carbon), i.e., $5\,.10^{11}$ C14 atoms per 12 g of carbon. When it dies, the C14 atoms begin to decay according to the law stated above, with $\mu = 1,2\,.10^{-4}$.
To estimate the age of an object of animal or vegetable origin, it is therefore sufficient to evaluate the number of C14 atoms contained in 12 g of carbon taken from this object.
a. We discover a vegetal remains containing $5\,.10^{10}$ C14 atoms per 12 g of carbon.
How old is he?
b. One calls period or half-life of a radioactive element the time necessary for the disintegration of half of the initial number of radioactive atoms.
Determine the half-life of carbon-14.

Correction

a. We have $N(t) = 5.10^{11}\, e^{1.2\,10^{-4}\,t} = 5.10^{10}$.

We deduce $$t = \frac{1}{1.2\,10^{-4}} \ln\left(\frac{5.10^{10}}{5.10^{11}}\right) = \frac{-\ln 10}{1.2\,10^{-4}} = -19\,188\ ans.$$

b. We have $N(t) = N_0\, e^{1.2\,10^{-4}\,t} = \frac{1}{2}N_0$.

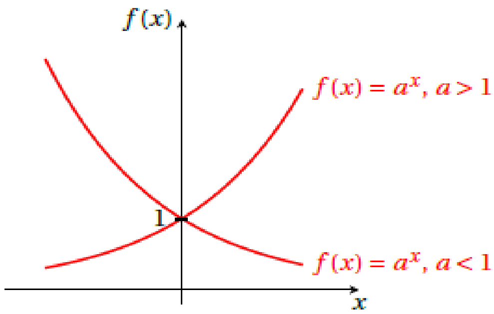
We deduce $$t = \frac{1}{1.2\,10^{-4}} \ln\left(\frac{1}{2}\right) = \frac{-\ln 2}{1.2\,10^{-4}} = -5\,776\ ans.$$

6.4. Basic exponential a>0

Definition:

For $a > 0$ *we define*

$\forall x \in \mathbb{R} \ : \quad a^x := exp(x\,\ln a)$.



Exercise

The Beer-Lambert law states that the amount of light I which penetrates to a depth of x meters in the ocean is given by $I(x) = I_0\, c^x$ with $0 < c < 1$ and I_0 is the amount of light at the surface.
a. Express x in terms of decimal logarithms.
b. Whether $c = 0.25$, calculate the depth at which $I = 0.01\, I_0$ (*this determines the area where photosynthesis can take place*).

Correction

a. We have $I = I_0\, c^x$, so $\frac{I}{I_0} = c^x$, we deduce $x = log_c\left(\frac{I}{I_0}\right) = \frac{\ln(I) - \ln(I_0)}{\ln(c)}$.

b. $c = 0.25$ and $I = 0.01\, I_0$ then $x = log_c\left(\frac{I}{I_0}\right) = log_c(0.01) = \frac{\ln(0.01)}{\ln(c)} \approx 3.32\ m$.

6.5. Power at real exponent

Definition:

For $\alpha \in \mathbb{R}$ *we define*

$$\forall x > 0\ :\quad x^\alpha := exp(\alpha \ln x)$$

Algebraic properties:

For every $x, y > 0\ ,\ a, b \in \mathbb{R}$*:*

1. $x^{a+b} = x^a \times x^b$, *2.* $(x^a)^b = x^{a\,b}$ *,3.* $x^{-b} = \frac{1}{x^b}$

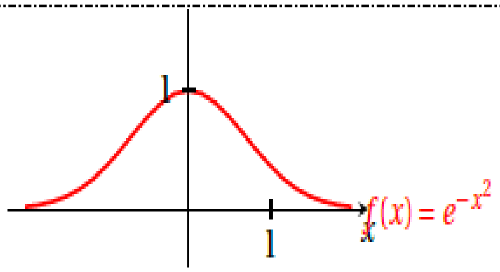
4. $(x\,y)^a = x^a \times y^a$, *5.* $\ln x^a = a \ln x$.

6.6. Gaussian functions.

Definition:

The Gaussian function is defined on $\mathbb{R}$ *by*

$$x \longrightarrow e^{-x^2}.$$



It is widely used in probability.

7 HYPERBOLIC AND INVERSE HYPERBOLIC FUNCTIONS

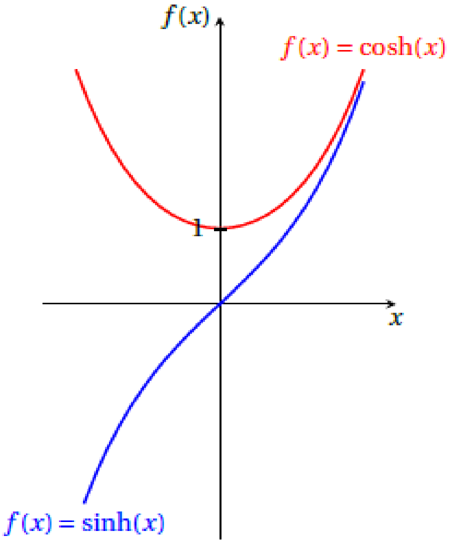
7.1. Hyperbolic functions

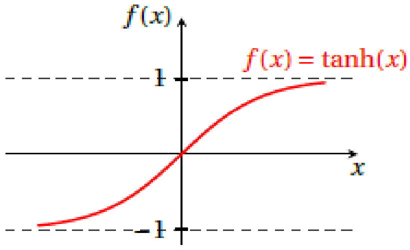
Definitions:

We define the functions **hyperbolic cosine**, **hyperbolic sine** *and* **hyperbolic tangent** *for every* $x \in \mathbb{R}$ *by, respectively*

$$\cosh(x) := \frac{e^x + e^{-x}}{2}, \qquad \sinh(x) := \frac{e^x - e^{-x}}{2},$$

$$\tanh(x) := \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1} = 1 - \frac{2}{1+e^{2x}}.$$





Properties:

- *Function* $\cosh$ *is* ***even*** *while the functions* $\sinh$ *and* $\tanh$ *are* ***odd.***

For every $x \in \mathbb{R}$ *we have*

- $\cosh(x) + \sinh(x) = e^x;\quad \cosh^2(x) - \sinh^2(x) = 1;\quad 1 - \tanh^2(x) = \dfrac{1}{\cosh^2(x)}.$

Derivation:

Functions $\cosh$, $\sinh$ *and* $\tanh$ *are differentiable on* $\mathbb{R}$, *we have:*

- $(\sinh(x))' = \cosh(x);\quad (\cosh(x))' = \sinh(x);$

- $(\tanh(x))' = \dfrac{1}{\cosh^2(x)} = 1 - \tanh^2(x).$

Exercise

Show that $\cosh^2 x - \sinh^2 x = 1.$

Correction

$$\cosh^2 x - \sinh x = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x + e^{-x})^2}{4} = \frac{e^{2x} + e^{-2x} + 2\,e^x\,e^{-x} - e^{2x} - e^{-2x} + 2\,e^x\,e^{-x}}{4}$$
$$= \frac{4\,e^x\,e^{-x}}{4} = 1$$

7.2. Inverse hyperbolic functions

Definitions:

Function **hyperbolic sine argument** *defined on* $\mathbb{R}$ *is the* ***inverse bijection of*** *the function* $\sinh$.

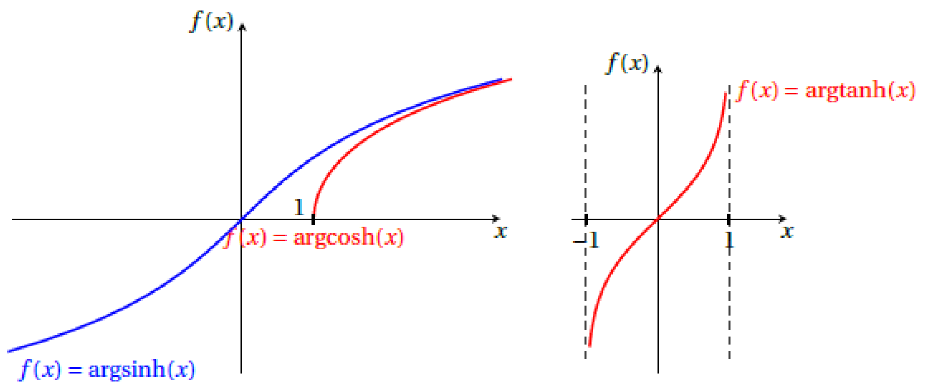
We note $argsinh$.

Function **hyperbolic cosine argument** *defined on* $]1, +\infty[$ *is the* ***inverse bijection of*** *the function* $\cosh$.

We note $argcosh$.

Function **hyperbolic tangent argument** *defined on* $]-1, +1[$ *is the* ***inverse bijection of*** *the function* $\cosh$.

We note $argcosh$.



Properties:

- *Functions* $\sinh$ *and* $\tanh$ *are* ***odd***.

For every $x \in \mathbb{R}$ *we have*

$\forall x \in \mathbb{R}$	$\text{argsinh}(x) = \ln(x + \sqrt{x^2+1})$
$\forall x \in [1, +\infty[$	$\text{argcosh}(x) = \ln(x + \sqrt{x^2-1})$
$\forall x \in]-1, 1[$	$\text{argtanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Derivation:

- *Function* $argcosh$ *is differentiable on* $]1, +\infty[$, *we have:*

$$(\text{argcosh}(x))' = \frac{1}{\sqrt{x^2 - 1}}.$$

- *Function* $argsinh$ *is differentiable on* $\mathbb{R}$, *we have:*

$$(\text{argsinh}(x))' = \frac{1}{\sqrt{x^2 + 1}}.$$

- *Function* $argcosh$ *is differentiable on* $]-1, +1[$, *we have:*

$$(\text{argtanh}(x))' = \frac{1}{1 - x^2}.$$

-

Reminder: $(\arccos x)' = \dfrac{-1}{\sqrt{1 - x^2}}.$

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$$

.$$(\arctan x)' = \frac{1}{\sqrt{1 + x^2}}$$