ALGERIAN REPUBLIC DEMOCRATIC AND POPULAR

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

UNIVERSITY CENTER NOUR BACHIR - EL BAYADH

Institute of Technology

Department of Electrical Engineering

Handout of Mathematics

for 1st year students Bachelor in
ST & SM

Analysis II

Presented by

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Analysis II

This course is intended for students in the first year of a Bachelor's degree in Engineering Sciences. Its objective is to provide basic mathematical tools for this sector.

Elementary numerical functions as well as equations and inequalities with a real variable correspond to the secondary school and are assumed to be known.

This manuscript is based on other books (partially, with or without modifications) like polycopids of Gloria Faccanoni, books of Herbert Amann and Joachim Escher in anlysis and others.

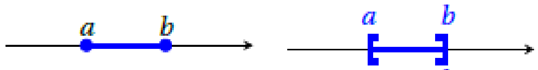
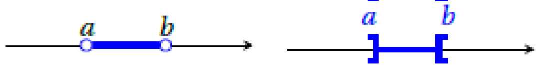
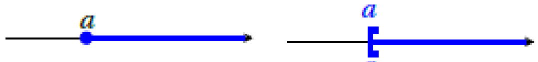
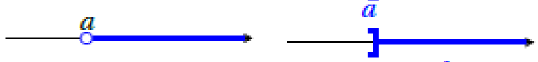
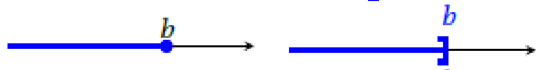
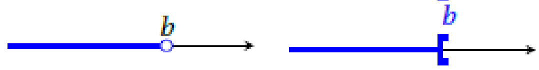
In this document are included many corrected exercises to show the interest and omnipresence of Mathematics in the various sciences (physics, economics, etc.).

Notations in Maths

Usual sets in mathematics

$\mathbb{N}$: set of natural numbers
$\mathbb{N}^*$: set of natural numbers without zero
$\mathbb{Z}$: set of relative numbers (positives, negatives or zero)
$\mathbb{Z}$: set of relative numbers without zero (positives or negatives)
$\mathbb{Q}$: set of rational numbers ($\frac{p}{q}$ such that $p \in \mathbb{Z}$ and $p \in \mathbb{N}^*$)
$\mathbb{R}$: set of real numbers
$\mathbb{R}^*$: set of natural numbers without zero
$\mathbb{R}$: set of complex numbers

Intervals

Inequalities	Corresponding set	Graphic representation
$a \leq x \leq b$	$[a, b]$	
$a < x < b$	$]a, b[$	
$a \leq x < b$	$[a, b[$	
$a < x \leq b$	$]a, b]$	
$x \geq a$	$[a, +\infty[$	
$x > a$	$]a, +\infty[$	
$x \leq b$	$]-\infty, b]$	
$x < b$	$]-\infty, b[$	

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Analysis Part

Chap 1 : Derivations and Approximations

1 DERIVATION

1.1. Definitions

Definition: (Derivation)

- *Let be $I \subset \mathbb{R}$ a non-empty open set. We say that a function $f : I \to \mathbb{R}$ is* ***differentiable at a point*** *x_0 (or admits a derivative at x_0) if the rate-increase $\frac{\Delta f(x)}{\Delta x} = \dfrac{f(x) - f(x_0)}{x - x_0}$ admits a limit $\Delta x \to 0$, noted $f'(x_0)$, when $\Delta x \to 0$:*

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

- *$f : I \to \mathbb{R}$ is* ***differentiable on I*** *if it is differentiable at any point of I.*
- *The function $x \in I \to f'(x)$ is called* ***derived function*** *of f and is denoted f' or (in Leibniz notation) $\dfrac{df}{dx}$.*

Theorem: (differentiability implies continuity)

Let f be a function defined on an ***open interval*** *$I \subset \mathbb{R}$* and $x_0 \in I$.

If f is ***differentiable*** *at x_0 then it is* ***continuous*** *at x_0.*

If f is ***differentiable*** *on I then it is* ***continuous*** *on I.*

Higher order derivatives

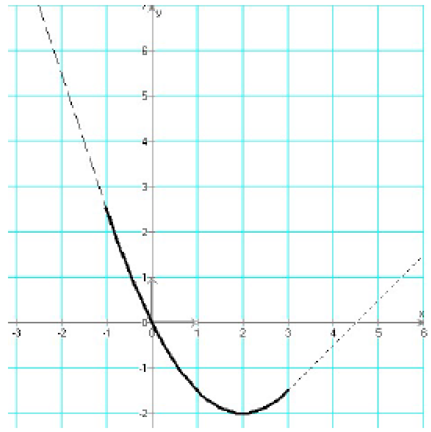
- *For $n \in \mathbb{N}$ we define by induction the* ***n-th derivative*** *(or* ***derivative of order n****) of f by setting $f^{(0)} = f$ then $f^{(n)} = (f^{(n-1)})'$.*
- *We say that f is* ***of class C^n on*** *I, and we write $f \in C^n(I)$, when f is n times differentiable on$I$ and the derivative $f^{(n)}$ is continuous on I.*
- *We say that f is* ***of class C^∞ on*** *I, and we write $f \in C^\infty(I)$, if f is of class C^n on I, for every $n \in \mathbb{N}$.*

Exercise

We want to extend a parabolic segment by two lines, so that the function obtained is everywhere derivable (*see the opposite drawing*).

Complete the formula below with equations of lines:

$$f(x)=\begin{cases}\dots\dots\dots & \text{for } x<-1\\ \dfrac{x^2}{2}-2x & \text{for } -1\le x\le 3\\ \dots\dots\dots & \text{for } x>3\end{cases}$$



Correction

We must look for real numbers a and b such that :

$$f(x)=\begin{cases}a(x+1)+2,5 & \text{for } x<-1\\ \dfrac{x^2}{2}-2x & \text{for } -1\le x\le 3\\ b(x-3)-1,5 & \text{for } x>3\end{cases}$$

Note that $x\to a(x+1)+2.5$ is differentiable on $]-\infty,1[$; $x\to\dfrac{x^2}{2}-2x$ is differentiable on $]-1,+3[$ and $x\to b(x-3)-1.5$ is differentiable on $]+3,+\infty[$.

It remains that f must be differentiable at points -1 and $+3$.

1) For $x_0=-1$:

Left derivative : $$\lim_{x\xrightarrow{x<-1}-1}\frac{f(x)-2.5}{x+1}=\lim_{x\xrightarrow{x<-1}-1}\frac{a(x+1)}{x+1}=a$$

Right derivative :

$$\lim_{x\xrightarrow{x>-1}-1}\frac{f(x)-2.5}{x+1}=\lim_{x\to-1}\frac{x^2/2-2x-5/2}{x+1}=\lim_{x\to-1}\frac{1}{2}\frac{x^2-4x-5}{x+1}=\lim_{x\to-1}\frac{1}{2}\frac{(x-5)(x+1)}{x+1}=\lim_{x\to-1}\frac{1}{2}(x-5)=-3$$

So we *must have* $a=-3.$

2) For $x_0=+3$:

Left derivative :

$$\lim_{x\overset{<}{\to}+3}\frac{f(x)+1.5}{x-3}=\lim_{x\to-1}\frac{x^2/2-2x+3/2}{x-3}=\lim_{x\to+3}\frac{1}{2}\frac{x^2-4x+3}{x-3}=\lim_{x\to-1}\frac{1}{2}\frac{(x-1)(x-3)}{x-3}=\lim_{x\to+3}\frac{1}{2}(x-1)=+1$$

Right derivative : $$\lim_{x\overset{>}{\to}+3}\frac{f(x)+1.5}{x+1}=\lim_{x\to+3}\frac{b(x-3)}{x-3}=b$$

Therefore, we *must* have $b=+1.$

We deduce

$$f(x)=\begin{cases}-3(x+1)+2,5 & \text{for } x<-1\\ \dfrac{x^2}{2}-2x & \text{for } -1\le x\le 3\\ +1(x-3)-1,5 & \text{for } x>3\end{cases}=\begin{cases}-3x-0,5 & \text{for } x<-1\\ \dfrac{x^2}{2}-2x & \text{for } -1\le x\le 3\\ x-4,5 & \text{for } x>3\end{cases}$$

Exercise

a. If a cube with sides of 2 cm increase by 1 cm/min, how does the volume increase?

b. If the area of a sphere with a radius of 10 cm increases by 5 cm2/min, how does the radius increase?

Correction

a) The volume of the cube with side x is $v = x^3$. We have $\frac{\Delta v}{\Delta t} \approx \frac{dv}{dt}$ (recall that $\frac{dv}{dt} := \lim_{t} \frac{\Delta v}{\Delta t}$) hence

$$\frac{\Delta v}{\Delta t} \approx \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} \approx 3x^2 \cdot \frac{\Delta x}{\Delta t} = 3(2)^2 \cdot 1 = 12\ cm^3/min.$$

b) The area of a sphere with a radius of 10 cm r is $s = 4\pi r^2$.

$\frac{\Delta s}{\Delta t} \approx \frac{ds}{dt} = \frac{ds}{dr} \cdot \frac{dr}{dt} \approx 8\pi r \cdot \frac{\Delta r}{\Delta t} = 5,$ we deduce $\frac{\Delta r}{\Delta t} = \frac{5}{8\pi r} = \frac{5}{8\pi\, 10} = \frac{1}{16\pi}\ cm/min.$

Exercise

A breach opened in the sides of a tanker. Suppose that the petrol extends around the breach according to a disc with a 2 m/s increasing radius. How fast does the surface of the oil slick-disc increase when the radius is 60 m?

Correction

Let *A* be the area of the disc (in m2), *r* the radius of the disc (in m) and *t* the time (in seconds) elapsed since the accident.

We want to calculate the rate of increase of the polluted area with respect to time, $\frac{\Delta A}{\Delta t} \approx \frac{dA}{dt}$ (remember $\frac{dA}{dt} := \lim_{t} \frac{\Delta A}{\Delta t}$).

We will use the relationship: $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$; the rate of increase of the radius is (given) $\frac{dr}{dt} = 2\ m/s.$

Consider the formula $A = \pi r^2$: Deriving with respect to *r*, we get: $\frac{dA}{dr} = 2\pi r$. So that, for $r = 60$ we'll get $\frac{dA}{dr} = 120\pi$

We deduce the variation of the speed of the surface of the oil spill when the radius of the slick is 60 m

$$\frac{\Delta A}{\Delta t} \approx \frac{dA}{dt} = 120\pi \cdot 2 \approx 754\ m^2/s.$$

1.2. Derivatives of usual functions

$(x^n)' = nx^{n-1}$

$(e^x)' = e^x$

$(a^x)' = a^x \ln(a)$

$(\ln(x))' = \frac{1}{x}$

$(\sin(x))' = \cos(x)$

$(\cos(x))' = -\sin(x)$

$(\tan(x))' = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$

$(\tan(x))' = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$

$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$

$(\arccos(x))' = -\frac{1}{\sqrt{1-x^2}}$

$(\arctan(x))' = \frac{1}{1+x^2}$

$(\sinh(x))' = \cosh(x)$

$(\cosh(x))' = \sinh(x)$

Examples

1) $\left(\frac{1}{x}\right)' = \left(x^{-1}\right)' = -1\,x^{-1-1} = -x^{-2} = \dfrac{-1}{x^2}$.

2) $\left(\sqrt{x}\right)' = \left(x^{1/2}\right)' = \frac{1}{2}\,x^{1/2-1} = \frac{1}{2}\,x^{-1/2} = \dfrac{-1}{2\sqrt{x}}$.

3) $\alpha \in \mathbb{R}\ :\ \left(x^{\alpha}\right)' = \left(e^{\alpha\ln x}\right)' = e^{\alpha\ln x} \times \alpha\,\dfrac{1}{x} = \alpha\,\dfrac{1}{x}\,x^{\alpha} = \alpha\,x^{\alpha-1}$.

4) $\left(2^x\right)' = \left(e^{x\ln 2}\right)' = e^{x\ln 2} \times \ln 2 = \ln 2\,.\,2^x$.

1.3. Calculation rules for derivatives

Derivable functions

- ***Elementary functions*** *such as polynomials, rational and irrational functions, exponential, logarithmic, trigonometric and hyperbolic functions* ***are differentiable in their respective domains.***

Derivative of compound functions

- *If f and u are differentiable then the* ***composite function*** $x \to (f \circ u)(x) = f\big(u(x)\big)$ *is* ***differentiable*** *on its domain and we have*

$$(f \circ u)'(x) = \left[f\big(u(x)\big)\right]' = u'(x) \times f'\big(u(x)\big).$$

or (in Leibniz notation easier to remember)

$$\frac{d(f \circ u)(x)}{dx} = \frac{d\left[f\big(u(x)\big)\right]}{dx} = \frac{d\,f\big(u\big)}{du}.\frac{du}{dx}.$$

Examples : on domain of U we have

1) $D_U = \{x \in \mathbb{R}\ :\ U(x) \neq 0\}$: $\quad \left(\dfrac{1}{U(x)}\right)' = \left(U^{-1}(x)\right)' = -1\,U'(x)\,U^{-2}(x) = \dfrac{-U'(x)}{U^2(x)}$.

2) $D_U = \{x \in \mathbb{R}\ :\ U(x) \geq 0\}$: $\quad \left(\sqrt{U(x)}\right)' = \left(U^{1/2}(x)\right)' = \dfrac{1}{2}\,U'(x)\,U^{\frac{1}{2}-1}(x) = \dfrac{U'(x)}{2\sqrt{U(x)}}$.

3) $D_U = \{x \in \mathbb{R}\ :\ U(x) > 0\}$: $\quad \alpha \in \mathbb{R}\ ,\ \left(U^{\alpha}(x)\right)' = \alpha\,U'(x)\,U^{\alpha-1}(x)$.

4) $D_U = \mathbb{R}$: $\quad \left(\sin U(x)\right)' = U'(x)\,\cos U(x)$.

Examples (derivatives of common composite functions)

$$\left([f(x)]^n\right)' = n[f(x)]^{n-1} f'(x) \qquad \left(\tan(f(x))\right)' = \frac{f'(x)}{\cos^2(f(x))} = 1+\tan^2(f(x))$$

$$\left(e^{f(x)}\right)' = e^{f(x)} f'(x) \qquad \left(\arcsin(f(x))\right)' = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$$

$$\left(a^{f(x)}\right)' = a^{f(x)} \ln(a) f'(x) \qquad \left(\arccos(f(x))\right)' = -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$$

$$\left(\ln(f(x))\right)' = \frac{f'(x)}{f(x)} \qquad \left(\arctan(f(x))\right)' = \frac{f'(x)}{1+(f(x))^2}$$

$$\left(\sin(f(x))\right)' = f'(x)\cos(f(x)) \qquad \left(\sinh(f(x))\right)' = f'(x)\cosh(f(x))$$

$$\left(\cos(f(x))\right)' = -f'(x)\sin(f(x)) \qquad \left(\cosh(f(x))\right)' = f'(x)\sinh(f(x))$$

Rules for calculating the derivative

- *The **sum**, **product** and **quotient**, of differentiable functions is a differentiable function over their domains of definition; and we have for differentiable functions f, g and $\lambda \in \mathbb{R}$:*

$$(f+g)' = f' + g' \ , \qquad (f \times g)' = f' \times g \ + \ f \times g' \quad ,$$

$$(\lambda \times f)' = \lambda \times f' \quad , \qquad \left(\frac{f}{g}\right)' = \frac{f' \times g \ - \ f \times g'}{g^2} \ (g(x) \neq 0).$$

- If f and g *are* n-times differentiable then the product $(f.g)$ is ***n-times differentiable*** and we have (***Leibniz formula***)

$$(f \cdot g)^{(n)} = f^{(n)} \cdot g + \binom{n}{1} f^{(n-1)} \cdot g^{(1)} + \cdots + \binom{n}{k} f^{(n-k)} \cdot g^{(k)} + \cdots + f \cdot g^{(n)} ,$$

which can be written $\qquad (f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} \cdot g^{(k)}.$

Examples

For n = 1 we'll get $\qquad (f \ . \ g)' = f' \ . \ g \ + \ f \ . \ g' \ ,$

for n = 2, we'll get $\qquad (f \ . \ g)'' = f'' \ . \ g \ + 2 \ f' \ . \ g' + f \ . \ g'' \ .$

Examples

Compute the n-th derivatives of $\exp(x).(x^2+1)$ for all n > 0.

Putting $f(x) = \exp(x)$ we get $f'(x) = \exp(x)$, $f''(x) = \exp(x)$, ...

Denote $g(x) = x^2+1$ then $g'(x) = 2x$, $g''(x) = 2$ and for k > 3, $g^{(k)}(x) = 0$.

Applying Leibniz's formula, we'll have

$$\begin{aligned}\exp(x).(x^2+1) = (f.g)^{(n)} &= f^{(n)}.g + f^{(n-1)}.g^{(1)} + f^{(n-2)}.g^{(2)} + f^{(n-3)}.0 + f^{(n-4)}.0 + \ldots \sin^2 x / \cos^2 x \\ &= \exp x.(x^2+1) + \exp x.(2x) + \exp x.(2) \\ &= (x^2+1)e^x + 2xe^x + 2e^x = (x^2+2x+3)e^x\end{aligned}$$

Derivative of the reciprocal bijection

- *If a bijection $f : E \to F$ is differentiable then its inverse bijection $f^{-1} : F \to E$ (defined by $y = f^{-1}(x) \iff x = f(y)$) is differentiable and we have*

$$\left(f^{-1}(x)\right)' = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{f'(y)} = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Notice.:

It is easier to find the formula by differentiating $f\left(g(x)\right) = x$ *with* $g = f^{-1}$ *:*

$$[f(g(x))]' = [x]' \iff g'(x) \times f'\left(g(x)\right) = 1 \iff \left(f^{-1}(x)\right)' = g'(x) = \frac{1}{f'\left(g(x)\right)} = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

2 FIRST-ORDER APPROXIMATION

2.1. Linearization - Differentiability

Definition: (differentiability)

*If a function f defined on an **open interval** $I \subset \mathbb{R}$* admits in a neighborhood of *a point $x_0 \in I$* an ***approximation of order 1*** *(or **linear**)* ***i.e.*** *that there exists a **linear map** $x \in V_{x_0} \to L(x)$ such as*

$$f(x) = L(x) + \circ(x - x_0);$$

*then we say that f is **differentiable** at the point x_0.*
*We also talk about **linearization** of the function f.*

NB: remember that $\circ(x - x_0) = (x - x_0)\,\epsilon(x - x_0)$ with $\epsilon(x - x_0) \xrightarrow[x \to x_0]{} 0$.

Theorem: (differentiability equivalent to differentiability)

*Let be f a function defined on an **open interval*** $I \subset \mathbb{R}$ and $x_0 \in I$.

f is **derivable** at x_0 ***i.e.*** $f'(x_0) := \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$ exists ***if and only if***

f is **differentiable** at x_0 ***i.e.*** *there is a linear map* $x \in V_{x_0} \to L(x)$ *such as* $f(x) = f(x_0) + L(x - x_0) + \circ(x - x_0)$.

We actually have $L(x - x_0) = f'(x_0)\,(x - x_0)$.

Indeed, the existence of the limit $f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$

is equivalent to one of the following two writings

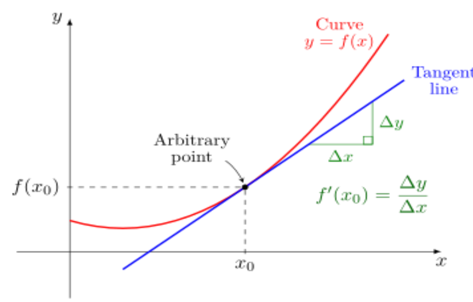
$f(x) = f(x_0) + f'(x_0)\,(x - x_0) + \circ(x - x_0)$ or $f(x_0 + h) = f'(x_0) \times h + f(x_0) + \circ(h)$.

Theorem: (linearization or approximation of order 1)

*If f is **differentiable (differentiable)** at x_0 then we can approximate $f(x)$ close to x_0.by a linear expression*

(approximation of order 1) :

$$f(x) \approx f(x_0) + f'(x_0)\,(x - x_0)$$



ATTENTION : Linearization depends on the point at which the function is linearized.

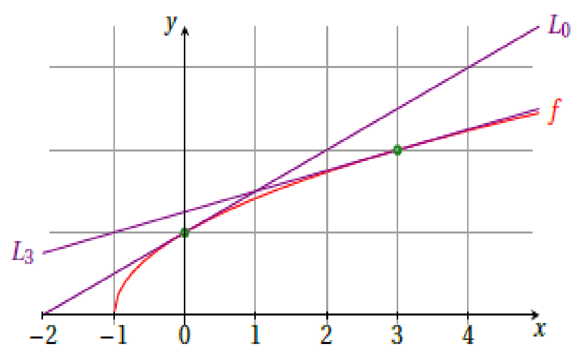
For example, linearizing the function $x \to f(x) = \sqrt{1+x}$ gives

Close to $x_0 = 0$

$f(x) \approx f(0) + f'(0)\,(x - 0) = 1 + \frac{1}{2}x$

Close to $x_0 = 3$

$f(x) \approx f(3) + f'(3)\,(x - 3) = \frac{5}{4} + \frac{1}{4}x$.



Example.1

Let $f(x) = (1+x)^n$, we have $f'(x) = n\,(1+x)^{n-1}$, linearization $f(x) \approx f(0) + f'(0)\,(x-0)$, we deduce

$$\boxed{(1+x)^n \approx 1 + n\,x \quad pour\ x << 1}$$

Simple formula to remember. It makes possible to calculate approximations of roots and powers of numbers close to unity. For examples :

$$\sqrt[3]{1.2} = (1+0.2)^{\frac{1}{3}} \approx 1 + \tfrac{1}{3}\,(0.2) \approx 1.066... \quad \text{(with calculator : } \sqrt[3]{1.2} = 1.062...\text{)}$$

$$(1.002)^{100} = (1+0.002)^{100} \approx 1 + 100\,(0.002) = 1.2$$

(with calculator : = $(1.002)^{100} = 1.22...$)

Example.2

Let $f(x) = \sin x$, linearization $\sin(x) \approx \sin(0) + \cos(0)\,(x-0)$, we deduce

$$\boxed{\sin(x) \approx x \quad pour\ x << 1}$$

This is the linearization that is performed to solve the pendulum equation in physics.

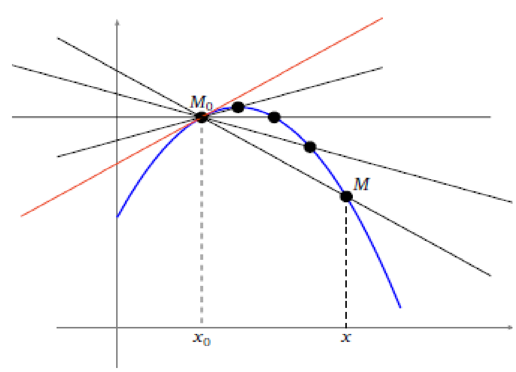
Line tangent to a point

The ***straight line*** which passes through the distinct points $(x_0, f(x_0))$ and $(x, f(x))$ has as ***slope coefficient*** $\dfrac{f(x)-f(x_0)}{x-x_0}$.

Taking the ***limit***, we find that the ***slope coefficient*** of the ***tangent*** is $f'(x_0)$.

An equation of the tangent at the point $(x_0, f(x_0))$ is then:

$$y = f'(x_0)\,(x-x_0) + f(x_0).$$



Exercise

The trajectory of an airplane in the opposite figure has the equation $y = \dfrac{2x+1}{x}$ *. The aircraft fires a laser beam along the tangent to its trajectory towards targets placed on the x'Ox axis at abscissa 1, 2, 3 and 4.*

a) *Will target no 4 be hit if the player shoots when the plane is at position (1, 3)?*

b) *Determine the abscissa of the plane allowing to reach the*

Correction

a) *Target no 4 will be hit if it is on the tangent to the curve at (1; 3).*

The derivative Is $y' = \dfrac{2\,.\,x - (2x+1)\,.\,1}{x^2} = \dfrac{-1}{x^2}$ *and the tangent equation is* $y = f'(1)(x-1) + 3 = -x + 4.$

For $x = 4$ *we have* $y = -4 + 4 = 0.$ *Therefore target no. 4 will be affected.*

b) *For target no. 2 to be hit, the tangent at (*x_0*, ?) of the aircraft trajectory must pass through target no. 2; therefore the couple* $(x; y) = (2; 0)$ *must verify the equation* $y = f'(x_0)(x - x_0) + f(x_0)$ ***i.e.*** $0 = \dfrac{-1}{x_0^2}(2 - x_0) + \dfrac{2x_0+1}{x_0}$,

that is $0 = \dfrac{-2+x_0}{x_0^2} + \dfrac{2x_0^2 + x_0}{x_0^2}$ *or again* $2x_0^2 + 2x_0 - 2 = 0.$ $\Delta' = 5,$ $x_1 = \dfrac{-1 \pm \sqrt{5}}{2}$.

One can deduce the abscissa of the plane-position making possible to reach target no. 2. is $x_0 = \dfrac{-1+\sqrt{5}}{2}$.

3 HIGHER-ORDER APPROXIMATION

3.1. Limited Taylor-Young expansion

Definition: (Limited development)

Let $a \in I$ *and* $n \in \mathbb{N}$*. We say that a function f admits a* ***limited expansion*** *(**LE**) to order n, at point a, if there are real numbers* $c_0, c_1, \ldots, c_n$ *such that for all x close enough to a we have:*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots + c_n(x-a)^n + o[(x-a)^n]$$

We recall that $\circ[(x-a)^n] = [(x-a)^n]\,\epsilon(x-a)$ *with* $\epsilon(x-a) \xrightarrow[x \to a]{} 0.$

✓ *The term* $c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots + c_n(x-a)^n$ *is called the* ***polynomial part*** *of the LE.*

✓ *The term* $\circ[(x-a)^n]$ *is the* ***rest*** *of the LE.*

✓ *The limited development (**LE**) if it exists is **unique** .*

✓ *If the function* f *is **even** (resp. **odd**) then the polynomial part of its LE at 0 contains only monomials of **even** (resp. **odd**) **degrees**.*

Theorem: (Taylor-Young formula)

Let f *be a function is of class* C^n *on* I *and* $a \in I$*. then for all* $x \in I$ *we have:*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + o[(x-a)^n]$$

*The **limited expansion** of* $f(x)$ *in the right-hand side of equality is called **Taylor-Young polynomials**.*

For n=1 : we find the approximation of order 1 (***linear***):

$$f(x) \approx f(a) + f'(a)(x-a).$$

For n=2 : we find the approximation of order 2 (***quadratic***):

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2.$$

Example

Let's look for various approximations of $f(x) = \exp x$ around the point $a = 0$:

✓ Approximation of order 1 (***linear***) :

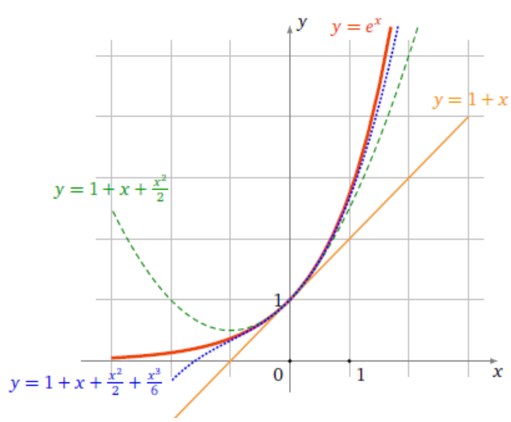
$f(x) \approx e^0 + e^0(x-0) = 1 + x$

✓ Approximation of order 2 (***quadratic***) :

$f(x) \approx e^0 + e^0(x-0) + \frac{e^0}{2!}(x-0)^2 = 1 + x + \frac{1}{2}x^2$

✓ Approximation of order 3 :

$f(x) \approx e^0 + e^0(x-0) + \frac{e^0}{2!}(x-0)^2 + \frac{e^0}{3!}(x-0)^3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}$



Example

For $f(x) = \ln(1+x)$ and we have: $f(0)=0$, $f'(x) = \dfrac{1}{1+x} \Longrightarrow f'(0)=1$,

$f''(x) = \dfrac{-1}{(1+x)^2} \Longrightarrow f'(0) = -1$, $f'''(x) = \dfrac{2}{(1+x)^3} \Longrightarrow f^3(0) = 2$, hence

✓ Approximation ***linear*** (of order 1) :

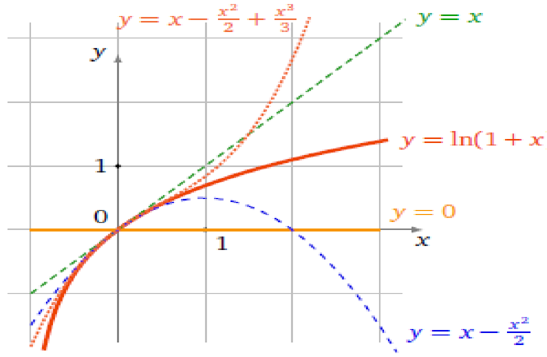
$f(x) \approx 0 + 1\,(x-0) = x$

✓ Approximation ***quadratic*** (d'ordre 2) :

$$f(x) \approx 0 + 1\,(x-0) + \frac{-1}{2!}(x-0)^2 = x - \frac{1}{2}x^2$$

✓ Approximation of order 3 :

$$f(x) \approx 0 + 1\,(x-0) + \frac{-1}{2!}(x-0)^2 + \frac{2}{3!}(x-0)^3 = 1 + x - \frac{1}{2}x^2 +$$



Note (important):

The equation of the tangent at the point of abscissa then a is

$$y = f(a) + f'(a)\,(x-a).$$

The quadratic approximation (of order 2) makes it possible to study the curvature *of the curve* of the function f

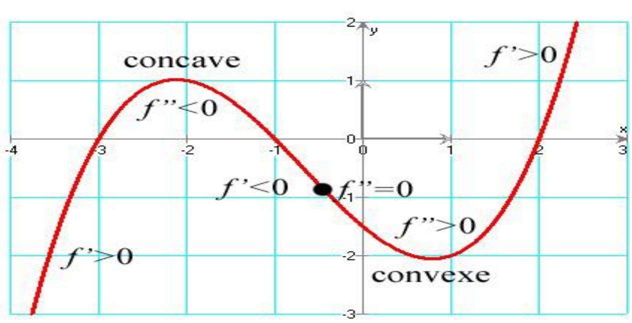
$$f(x) \approx f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}(x-a)^2 \iff f(x) - y \approx \frac{f''(a)}{2!}(x-a)^2.$$

So, on an interval I we have:

✓ ***If $f''<0$ then the curve of f is below the tangent: concave function.***

✓ ***If $f''>0$ then the curve of f is above the tangent: function convex.***

*The point where there is a change in curvature is called **the inflection point**. To determine it analytically, it is necessary to solve the equation $f''(x)=0$ and then search among the solutions for those where f'' changes the sign.*



Theorem: (Error of the approximation)

*If a function f is $n+1$ differentiable and P_n is its Taylor polynomial of order n generated by f at $a \in I$, if $\left|f^{(n+1)}(x)\right|$ is bounded over I by a real M **i.e.** $\left|f^{(n+1)}(x)\right| \leq M$, then $\forall x \in I$:*

$$\left|f(x) - P_n(x)\right| \leq \frac{|x-a|^{n+1}}{(n+1)!} M.$$

Example

The linearization close to $a = 0$ of $f(x) = \sin x$ gives $\sin(x) \approx x$.

What is the precision of this approximation if $|x| \leq 0.5$ **i.e.** $x \in [-0.5, +0.5]$?

We have $\max_{|x|\leq 0.5} |f''(x)| = \max_{|x|\leq 0.5} |-\sin(x)| = \sin(0.5)$ *we deduce*

$$\forall x \in [-0.5, +0.5] \quad : \quad \left|f(x) - P_1(x)\right| = \left|\sin(x) - x\right| \leq \frac{(0.5)^2}{2!} \sin(0.5) < 0.06.$$

3.2. LE at the origin of usual functions

We have to retain the following LE at 0 of usual functions:

$$\boxed{\exp x = 1 + \tfrac{x}{1!} + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots + \tfrac{x^n}{n!} + x^n \epsilon(x)}$$

$$\operatorname{ch} x = 1 + \tfrac{x^2}{2!} + \tfrac{x^4}{4!} + \cdots + \tfrac{x^{2n}}{(2n)!} + x^{2n+1}\epsilon(x)$$

$$\operatorname{sh} x = \tfrac{x}{1!} + \tfrac{x^3}{3!} + \tfrac{x^5}{5!} + \cdots + \tfrac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\epsilon(x)$$

$$\cos x = 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \cdots + (-1)^n \tfrac{x^{2n}}{(2n)!} + x^{2n+1}\epsilon(x)$$

$$\sin x = \tfrac{x}{1!} - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \cdots + (-1)^n \tfrac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\epsilon(x)$$

$$\boxed{\ln(1+x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \cdots + (-1)^{n-1} \tfrac{x^n}{n} + x^n \epsilon(x)}$$

$$\boxed{(1+x)^\alpha = 1 + \alpha x + \tfrac{\alpha(\alpha-1)}{2!} x^2 + \cdots + \tfrac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n + x^n \epsilon(x)}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + x^n \epsilon(x)$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + x^n \epsilon(x)$$

$$\sqrt{1+x} = 1 + \tfrac{x}{2} - \tfrac{1}{8}x^2 + \cdots + (-1)^{n-1} \tfrac{1\cdot 1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!} x^n + x^n \epsilon(x)$$

Important remarks :

- *The LE of* $\cosh x$ *is the even part of the DL of* $\exp x$ *(we retain the monomials of even degree).*
- *The LE of* $\sinh x$ *is the odd part of the DL of* $\exp x$ *(we retain only the odd degrees).*
- *The LE of* $\cos x$ *is the even part of the DL of* $\exp x$ *by alternating the sign* $+$*and* $-$.
- *The LE* $\sin x$ *is the odd part of* $\exp x$ *by alternating the signs* $+$*and* $-$.
- *For* $\ln(1+x)$ *there is no constant term, no factorial and the signs alternate.*

3.3. LE of functions at any point

The function f admits a LE close to a point $x = a$ **if and only** if the function $t \to f(t+a)$ admits a LE close to $x = 0$.

Therefore, we reduce the problem to 0 by the change of variables $t = x - a$.

Examples.

1. LE of $f(x) = e^x$ at $a = 1$.

We pose $t = x - 1$. If *x* is close to 1 then *t* is close to 0.

We will look for a LE of e^t near $t = 0$.

$$e^x = e^{t+1} = e\,e^t = e\,[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+...+\frac{t^n}{n!}+o(t^n)] = e\,[1+(x-1)+\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+...+\frac{(x-1)^n}{n!}+o[(x-1)^n]]$$

So close to $a = 1$ we get

$$e^x = e\,[1+(x-1)+\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+...+\frac{(x-1)^n}{n!}+o[(x-1)^n]].$$

2. LE of $g(x) = \sin x$ close to $a = \pi/2$. We pose $t = x - \pi/2$, we have $x \to \pi/2 \iff t \to 0$.

$$\begin{aligned}\sin x = \sin(t+\pi/2) = \cos(t) &= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - ... + (-1)^n \frac{t^{2n}}{(2n)!} + o(t^{2n+1}) \\ &= 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - ... + (-1)^n \frac{(x-\pi/2)^{2n}}{(2n)!} + o[(x-\pi/2)^{2n+1}]\end{aligned}$$

3. LE of $h(x) = \ln(1+3x)$ at a=1 to order 3.

We set $t = x - 1$, we have $x \to 1 \iff t \to 0$.

$$\ln(1+3x) = \ln(1+3(t+1)) = \ln(4+3t) = \ln 4(1+\frac{3t}{4}) = \ln 4 + \ln(1+\frac{3t}{4}).$$

We pose $T = \frac{3t}{4}$, we have $x \to 1 \iff t \to 0 \iff T \to 0$; we use

$$\ln(1+T) = T - \tfrac{T^2}{2} + ... + (-1)^{n-1}\tfrac{T^n}{n} + o(T^n).$$

$$h(x) = \ln(1+3x) = \ln 4 + \ln(1+\frac{3t}{4}) = \ln 4 + \ln(1+T) = \ln 4 + T - \frac{T^2}{2} + \frac{T^3}{3} + o(T^3)$$

$$= \ln 4 + \frac{3t}{4} - \frac{\left(\frac{3t}{4}\right)^2}{2} + \frac{\left(\frac{3t}{4}\right)^3}{3} + o((\frac{3t}{4})^3) = \ln 4 + \frac{3}{4}t - \frac{9}{32}t^2 + \frac{9}{64}t^3 + o[t^3]$$

$$= \ln 4 + \frac{3}{4}(x-1) - \frac{9}{32}(x-1)^2 + \frac{9}{64}(x-1)^3 + o[(x-1)^3] .$$

3.4. Operations on limited expantion

Let f and g be two functions which admit LEs at 0 to order n :

$$f(x) = c_0 + c_1 x + ... + c_n x^n + o(x^n) := P_f + o(x^n),$$

$$g(x) = d_0 + d_1 x + ... + d_n x^n + o(x^n) := P_g + o(x^n).$$

> ***Theorem: (Sum and product)***
>
> - *The TAYLOR polynomial of order n generated for the* ***sum*** $f+g$ *is the* ***polynomial sum*** $P_f + P_g$;
> $$(f+g)(x) = (c_0+d_0) + (c_1+d_1)\,x + ... + (c_n+d_n)\,x_n + o(x^n).$$
> - *The TAYLOR polynomial of order n generated for the* ***product*** $f.g$ *is the* ***polynomial product*** $P_f \times P_g$ ***truncated*** *to order n,* ***i.e.*** *that we keep only the monomials of degree ≤ n ;*

Example.

We have the LE of order 2:

$\cos x = 1 - \frac{1}{2}x^2 + o(x^2)$ and $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)$ then:

$$\cos x + \sqrt{1+x} = [1 - \frac{1}{2}x^2 + o(x^2)] + [1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)]$$

$$= (1+1) + \frac{1}{2}x + (-\frac{1}{2} - \frac{1}{8})\,x^2 + o(x^2) \;=\; 2 + \frac{1}{2}x - \frac{5}{8}x^2 + o(x^2)$$

$$\cos x\,\sqrt{1+x} = [1 - \frac{1}{2}x^2 + o(x^2)] \times [1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)]$$

$$= 1 \times [1 + \frac{1}{2}x - \frac{1}{8}x^2] - \frac{1}{2}x^2 \times 1 + o(x^2) + o(x^2) \quad = 1 + \frac{1}{2}x - \frac{5}{8}x^2 + o(x^2)$$

Theorem: (Composition)

- *If $g(0)=0$ then the **composite** function $f\circ g$ admits a LE of order n at a=0 whose polynomial part is the truncated polynomial at order n of the composite $P_f[P_g(x)]$.*

Examples:

1) **Calculation of the LE of $h(x)=\sin\ln(1+x)$ at 0 to order 3.**

We put here $f(u)=\sin u$ and $u=g(x)=\ln(1+x)$. We have

$$(f\circ g)(x)=f[g(x)]=f(u)=\sin u=\sin\ln(1+x) \qquad \text{and } g(0)=0.$$

The LEs: $\sin u = u-\frac{1}{3!}u^3+o(u^3)$ and $u=\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+o(x^3)$,

so $u^2=[x-\frac{1}{2}x^2+\frac{1}{3}x^3+o(x^3)]^2=x^2-2\,x\,\frac{1}{2}x^2+o(x^3)=x^2-x^3+o(x^3)$

and $u^3=[x-\frac{1}{2}x^2+\frac{1}{3}x^3+o(x^3)]\times[x^2-x^3+o(x^3)]=x^3+o(x^3)$.

Consequently

$$\sin\ln(1+x)=\sin u=u-\frac{1}{3!}u^3+o(u^3) = [x-\frac{1}{2}x^2+\frac{1}{3}x^3+o(x^3)]-\frac{1}{6}[x^3+o(x^3)]+o(x^3)$$
$$=x-\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3)$$

2) **Calculation of the LE of $h(x)=\sqrt{\cos x}$ near 0 to order 4.**

We know the LEs: $\cos x=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+o(u^4)$ and $\sqrt{1+u}=1+\frac{1}{2}u-\frac{1}{8}u^2+o(x^3)$

We put $f(u)=\sqrt{u}$ and $g(x)=1+u=\cos(x)$. We have

$$h(x)=(f\circ g)(x)=f[g(x)]=f(1+u)=\sqrt{1+u} \qquad \text{and} \quad g(0)=0.$$

$u=\cos x-1=[1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+o(u^4)]-1=-\frac{1}{2}x^2+\frac{1}{24}x^4+o(u^4)$

$u^2=[-\frac{1}{2}x^2+\frac{1}{24}x^4+o(u^4)]^2=\frac{1}{4}x^4+o(x^4)$.

We deduce

$$\sqrt{\cos x}=\sqrt{1+u}=1+\frac{1}{2}u-\frac{1}{8}u^2+o(x^4) = 1+\frac{1}{2}[-\frac{1}{2}x^2+\frac{1}{24}x^4+o(u^4)]-\frac{1}{8}[\frac{1}{4}x^4+o(x^4)]+o(x^4)$$
$$=1-\frac{1}{4}x^2+\frac{1}{48}x^4-\frac{1}{32}x^4+o(x^4)=1-\frac{1}{4}x^2-\frac{1}{96}x^4+o(x^4)$$

Theorem: (Division)

- *By carrying out the* ***division according to the <u>increasing</u> powers*** *of P_f by P_g to the order n we will obtain the writing:*

$$P_f = P_g\, Q + x^{n+1}\, R \qquad \text{with } deg\, Q \leq n.$$

Then ***Q is the polynomial part of the LE*** *at 0 to order n of* $\dfrac{P(x)}{Q(x)}$.

<u>Examples.</u>

Find the LE of $\dfrac{2+x+2x^2}{1+x^2}$ to order 2.

$$\begin{array}{l|l} 2+x+2x^3 & 1+x^2 \\ \cline{2-2} 2+2x^2 & 2+x-2x^2 \\ \cline{1-1} x-2x^2+2x^3 & \\ x+x^3 & \\ \cline{1-1} -2x^2+x^3 & \\ -2x^2-2x^4 & \\ \cline{1-1} x^3+2x^4 & \end{array}$$

We deduce $$\frac{2+x+2x^2}{1+x^2} = 2+x-2x^2+o(x^2)$$

3.5. Applications of LEs

3.5.1. Limit calculations:

1) Calculate $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{e^x - 1}$. Let's use the LEs:

$$e^x = 1 + x + \frac{t^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + o(x^n)\,;\cos x = 1 - \frac{t^2}{2!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots + \frac{x^n}{n!} + o(x^{2n}).$$

$$\frac{\cos x - 1}{e^x - 1} = \frac{1 - \frac{1}{2}\,x^2 + o(x^2) - 1}{1 + x - \frac{1}{2}\,x^2 + o(x^2) - 1} = \frac{-\frac{1}{2}\,x^2 + o(x^2)}{x - \frac{1}{2}\,x^2 + o(x^2)} = \frac{-x^2 + o(x^2)}{2x - x^2 + o(x^2)}.$$

$$
\begin{array}{l|l}
-x^2 & 2x - x^2 \\ \cline{2-2}
-x^2 + \frac{1}{2}\,x^3 & -\frac{1}{2}\,x \\ \hline
-\frac{1}{2}\,x^3 &
\end{array}
$$

We deduce $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{e^x - 1} = \lim\limits_{x\to 0} -\dfrac{1}{2}\,x + o(x) = 0$.

2) **Calculate** $\lim\limits_{x\to 0} \dfrac{\ln(1+x) - \tan x + \frac{1}{2}\sin^2 x}{3x^2\, sin^2 x} := \dfrac{f(x)}{g(x)}$. We recall:

$\ln(1+x) = x - \dfrac{t^2}{2} + ... + (-1)^{n-1}\dfrac{x^n}{n} + o(x^n)$; $\sin x = x - \dfrac{x^3}{3!} + ... + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$

$\cos x = 1 - \dfrac{x^2}{2!} + ... + (-1)^n \dfrac{x^{2n}}{(2n)!} + o(x^{2n})$; $\tan x = \dfrac{\sin x}{\cos x} = \dfrac{x - \dfrac{x^3}{3!} + ...}{1 - \dfrac{x^2}{2!} + ...}$.

$$
\begin{array}{l|l}
x - \frac{1}{6}\,x^3 + ... & 1 - \frac{1}{2}\,x^2 \\ \cline{2-2}
x - \frac{1}{2}\,x^3 & x + \frac{1}{3}\,x^3 + ... \\ \cline{1-1}
+\frac{1}{3}\,x^3 + ... & \\
+\frac{1}{3}\,x^3 - \frac{1}{6}\,x^5 & \\ \cline{1-1}
\ldots &
\end{array}
$$

We deduce

$$
\begin{aligned}
f(x) &:= \ln(1+x) - \tan x + \frac{1}{2}\sin^2 x \\
&= [x - \frac{1}{2}\,x^2 + \frac{1}{3}\,x^3 - \frac{1}{4}\,x^4 + o(x^4)] - [x + \frac{1}{3}\,x^3 + o(x^4)] + \frac{1}{2}[x - \frac{1}{3!}\,x^3 + o(x^4)]^2 \\
&= -\frac{1}{2}\,x^2 - \frac{1}{4}\,x^4 + o(x^4) + \frac{1}{2}[x^2 - 2\,x\,\frac{1}{3!}\,x^3 + o(x^4)] \\
&= -\frac{1}{2}\,x^2 - \frac{1}{4}\,x^4 + \frac{1}{2}x^2 - \frac{1}{6}\,x^4 + o(x^4) = -\frac{5}{12}\,x^4 + o(x^4)\ .
\end{aligned}
$$

$$g(x) := 3x^2\,\sin^2 x \;=\; 3x^2\,[x - \frac{1}{3!}\,x^3 + o(x^4)]^2$$
$$= 3x^2\,[x^2 - 2\,x\,\frac{1}{3!}\,x^3 + o(x^4)] \;=\; 3x^4 + o(x^4)\ .$$

Then $$\lim_{x\to 0}\frac{\ln(1+x)-\tan x+\frac{1}{2}\sin^2 x}{3x^2\,sin^2 x} = \lim_{x\to 0}\frac{-\frac{5}{12}\,x^4+o(x^4)}{3x^4+o(x^4)} = -\frac{5}{36}.$$

NB: by calculating the LE at a lower order, we could not have concluded, because we would have obtained $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{o(x^4)}{o(x^4)}$ which remains an indeterminate form.

3.5.2. Equivalences:

1) Give simple equivalents close to 0 for the following functions:

a) $2e^x - \sqrt{1+4x} - \sqrt{1+6x^2}$ **b)** $(\cos x)^{\sin x} - (\cos x)^{\tan x}$.

It is a question of determining the first terms of the LEs.

a) $2e^x - \sqrt{1+4x} - \sqrt{1+6x^2} := f(x)$. We have the LEs:

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n)$ and

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}\,x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{2!}\,x^3 + \dots + \frac{\alpha(\alpha-1)..(\alpha-n+1)}{n!}\,x^n + o(x^n),$$

then for $\alpha = \frac{1}{2}$

$$(1+x)^{\frac{1}{2}} = \sqrt{1+x} = 1 + \tfrac{1}{2}x + \tfrac{1(-1)}{2^2\ 2!}\,x^2 + \tfrac{1(-1)(-3)}{2^3\ 3!}\,x^3 + \dots + (-1)^{n-1}\tfrac{1(1)(3)(5)..(2n-3)}{2^n\ n!}\,x^n + o(x^n).$$

To order 3 we'll have $(1+x)^{\frac{1}{2}} = \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}\,x^2 + \frac{1}{16}\,x^3 + +o(x^3).$

We deduce

$$f(x) = [2 + 2x + x^2 + \frac{x^3}{3} + o(x^3)] - [1 + \frac{1}{2}(4x) - \frac{1}{8}\,(4x)^2 + \frac{1}{16}\,(4x)^3 + o(x^3)] - [1 + \frac{1}{2}(6x^2) + o(x^3)]$$
$$= +x^2 + \frac{x^3}{3} + \frac{16}{8}\,x^2 - \frac{4^3}{16}\,x^3 - \frac{6}{2}x^2 + o(x^3) = +\frac{x^3}{3} - 4\,x^3 + o(x^3) = -\frac{11}{3}\,x^3 + o(x^3)$$

So close to 0 we have $$2e^x - \sqrt{1+4x} - \sqrt{1+6x^2} \sim -\frac{11}{3}\,x^3.$$

b) $g(x) := (\cos x)^{\sin x} - (\cos x)^{\tan x} = \exp[\sin x \ \ln\cos x] - \exp[\tan x \ \ln\cos x]$.

We know the LEs: $\sin x = x - \dfrac{x^3}{3!} + ... + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$

$\cos x = 1 - \dfrac{x^2}{2!} + ... + (-1)^n \dfrac{x^{2n}}{(2n)!} + o(x^{2n})$; $\tan x = \dfrac{\sin x}{\cos x} = x + \frac{1}{3}x^3 + o(x^4)$

$\ln(1-x) = x + \frac{1}{2}x^2 + ... + \frac{1}{n}x^n + o(x^n)$ and $e^x = 1 + x + \dfrac{t^2}{2!} + \dfrac{x^3}{3!} + ... + \dfrac{x^n}{n!} + o(x^n)$.

We deduce

$$\begin{aligned}
g(x) &= \exp[(\sin x)\ \ln(\cos x)] - \exp[(\tan x)\ \ln(\cos x)] \\
&= \exp\left[\left(x - \frac{1}{3!}x^3 + o(x^3)\right) \ln\left(1 - \frac{x^2}{2!} + o(x^3)\right)\right] - \exp\left[\left(x + \frac{1}{3}x^3 + o(x^3)\right) \ln\left(1 - \frac{x^2}{2!} + o(x^3)\right)\right] \\
&= \exp\left[\left(x - \frac{1}{3!}x^3 + o(x^3)\right)\left(\frac{x^2}{2} + \frac{1}{2}[\frac{x^2}{2}]^2 + o(x^3)\right)\right] - \exp\left[\left(x + \frac{1}{3}x^3 + o(x^3)\right)\left(\frac{x^2}{2} + \frac{1}{2}[\frac{x^2}{2}]^2 + o(x^3)\right)\right] \\
&= \exp\left[\frac{x^3}{2} + \frac{1}{2}\frac{x^5}{4} - \frac{1}{3!}\frac{x^5}{2} + o(x^5)\right] - \exp\left[\frac{x^3}{2} + \frac{1}{2}\frac{x^5}{4} + \frac{1}{3}\frac{x^5}{2} + o(x^5)\right] \\
&= 1 + \frac{x^3}{2} + \frac{1}{2}\frac{x^5}{4} - \frac{1}{6}\frac{x^5}{2} + o(x^5) - 1 - \frac{x^3}{2} - \frac{1}{2}\frac{x^5}{4} - \frac{1}{3}\frac{x^5}{2} + o(x^5) \\
&= -\frac{3}{6}\frac{1}{2}x^5 + o(x^5) = -\frac{1}{4}x^5 + o(x^5)
\end{aligned}$$

So close to 0 we have $\quad (\cos x)^{\sin x} - (\cos x)^{\tan x} \sim -\dfrac{1}{4}x^5$.

2) Give an equivalent close to $+\infty$ of $\quad \sqrt{x^2+1} - 2\sqrt[3]{x^3+x} + \sqrt[4]{x^4+x^2}$.

Reminder: $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{2!}x^3 + ... + \frac{\alpha(\alpha-1)..(\alpha-n+1)}{n!}x^n + o(x^n)$

then close to 0 we have

$$(1+x)^{\frac{1}{2}} = \sqrt{1+x} = 1 + \tfrac{1}{2}x + \tfrac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + o(x^2) = 1 + \tfrac{1}{2}x - \tfrac{1}{2^2\ 2!}x^2 + o(x^2)$$

$$(1+x)^{\frac{1}{3}} = 1 + \tfrac{1}{3}x + \tfrac{\frac{1}{3}(\frac{1}{3}-1)}{2!}x^2 + o(x^2) = 1 + \tfrac{1}{3}x - \tfrac{2}{3^2\ 2!}x^2 + o(x^2)$$

$$(1+x)^{\frac{1}{4}} = 1 + \tfrac{1}{4}x + \tfrac{\frac{1}{4}(\frac{1}{4}-1)}{2!}x^2 + o(x^2) = 1 + \tfrac{1}{4}x - \tfrac{3}{4^2\ 2!}x^2 + o(x^2)$$

Noticing that $x \to +\infty \iff \dfrac{1}{x} \to 0$, we deduce that close to $+\infty$ we have

$$\sqrt{x^2+1} = x\ \sqrt{1 + \left(\tfrac{1}{x^2}\right)} = x\ \left[1 + \tfrac{1}{2}\tfrac{1}{x^2} - \tfrac{1}{8}\left(\tfrac{1}{x^2}\right)^2 + o(\tfrac{1}{x^4})\right] = x + \tfrac{1}{2}\tfrac{1}{x} - \tfrac{1}{8}\tfrac{1}{x^3} + o(\tfrac{1}{x^4})$$

$$-2\sqrt[3]{x^3+x} = -2x\ \sqrt[3]{1 + \left(\tfrac{1}{x^2}\right)} = -2x\ \left[1 + \tfrac{1}{3}\left(\tfrac{1}{x^2}\right) - \tfrac{1}{9}\left(\tfrac{1}{x^2}\right)^2 + o(\tfrac{1}{x^4})\right] = -2x - \tfrac{2}{3}\tfrac{1}{x} + \tfrac{2}{9}\tfrac{1}{x^3} + o(\tfrac{1}{x^4})$$

$$\sqrt[4]{x^4+x^2} = x\ \sqrt[4]{1+\left(\frac{1}{x^2}\right)} = x\ \left[1+\frac{1}{4}\frac{1}{x^2}-\frac{3}{32}\left(\frac{1}{x^2}\right)^2+o(\frac{1}{x^4})\right] = x+\frac{1}{4}\frac{1}{x}-\frac{3}{32}\frac{1}{x^3}+o(\frac{1}{x^4})$$

Adding these results, we get

$$\begin{aligned} f(x) &:= \sqrt{x^2+1}-2\sqrt[3]{x^3+x}+\sqrt[4]{x^4+x^2} \\ &= [x+\frac{1}{2}\frac{1}{x}-\frac{1}{8}\frac{1}{x^3}+o(\frac{1}{x^4})]+[-2x-\frac{2}{3}\frac{1}{x}+\frac{2}{9}\frac{1}{x^3}+o(\frac{1}{x^4})]+[x+\frac{1}{4}\frac{1}{x}-\frac{3}{32}\frac{1}{x^3}+o(\frac{1}{x^4})] \\ &= \frac{1}{12}\frac{1}{x}+o(\frac{1}{x}) \end{aligned}$$

So close to $+\infty$ $\qquad \sqrt{x^2+1}-2\sqrt[3]{x^3+x}+\sqrt[4]{x^4+x^2} \sim \frac{1}{12}\frac{1}{x}$.

3.5.3. Others :

Find the tangent of the graph, at point of abscissa $a = 1/2$, of a function f *defined by* $f(x) = x^4 - 2x^3 + 1$; and specify the position of the graph with respect to the tangent

Let's use the LE of $f(x)$ at point $a = \frac{1}{2}$. $f'(x) = 4x^3 - 6x^2$, $f''(x) = 12x^2 - 12x$; then

$$f(x) = f(1/2)+f'(1/2)(x-1/2)+\frac{f''(1/2)}{2!}(x-1/2)^2+o[(x-1/2)^2] = \frac{13}{36}-(x-1/2)-\frac{3}{2}(x-1/2)^2+o[(x-1/2)^2]$$

We deduce the equation of the tangent $y = \frac{13}{36} - (x - 1/2)$.

The position of the graph with respect to the tangent depends on the sign of

$$f(x) - y = -\frac{3}{2}(x-1/2)^2 + o[(x-1/2)^2]$$

which is negative; this means that the graph is below the tangent.

Chap 2 : Integration

1 Primitives

Definition: (primitive)

*Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ admits a **primitive** if there exists a differentiable function F such that*

$$\forall x \in I, F'(x) = f(x)$$

*F is called **primitive** of f .*

Proposition (Existence of primitives)

*If a function $f : I \to \mathbb{R}$ is **continuous** then f has a **primitive**.*

Properties

- ✓ *If F is a **primitive** of f then, for any real c , the function $F + c$ is **also a primitive** of f (an infinity of primitives).*
- ✓ *Any primitive of f is necessarily of the form $F + c$ for some real constant c.*

Notation : The set of primitives of a function *f* is denoted by $\int f(x)\, dx$ (***indefinite integral*** *of f*).

Remarks :

1) The variable x in $\int f(x)\, dx$ is mute in the sense that it can be changed:

$$\int f(t)dt = \int f(z)dz = .. = \int f(\blacksquare)d\blacksquare$$

2) If $a \in I$ then $F(x) = \int_a^x f(t)\, dt$ is the only primitive that vanishes at a.

1.1. Common function primitives

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \text{ pour } n \neq -1$$

$$\int \frac{1}{x} \, dx = \ln(x) + c$$

$$\int \sin(x) \, dx = -\cos(x) + c$$

$$\int \cos(x) \, dx = \sin(x) + c$$

$$\int \frac{1}{\cos^2(x)} \, dx = \tan(x) + c$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin(x) + c = -\arccos(x) + c$$

$$\int e^x \, dx = e^x + c$$

$$\int a^x \, dx = (\log_a e) a^x + c$$

$$\int \frac{1}{1+x^2} \, dx = \arctan(x) + c$$

$$\int \cosh(x) \, dx = \sinh(x) + c$$

$$\int \sinh(x) \, dx = \cosh(x) + c$$

$$\int \frac{1}{\cosh^2(x)} \, dx = \tanh(x) + c$$

2 INTEGRATION

Fundamental formula of integral calculus :

*If a function f is **continuous** on the interval $[a, b]$ and if F is a **primitive** of f $(F' = f)$, then*

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

Furthermore, for every $x \in (a, b)$,

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x).$$

N.B.: *In practice, an integral is the continuous analogue of a summation of infinitesimal quantities.*

2.1. Properties

Properties :

*Let f, g: be two **integrable functions** on $[a, b]$, then*

✓ ***Linearity :*** $\forall \alpha, \beta \in \mathbb{R}$:

$$\int_a^b (\alpha f(x) + \beta g(x)) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx$$

✓ **Positivity :** *If* $\forall x \in [a,b] \; : \; f(x) \leq g(x)$ *then*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

✓ **Absolute value :** *we* ***always*** *have*

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

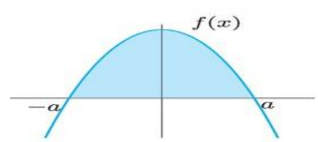
✓ **Chasles's relation :** *for every* $c \in [a,b]$*:*

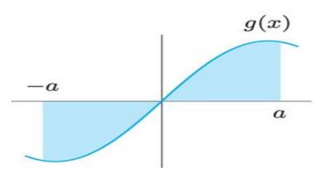
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In particular $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

✓ **Parity:** *for all* $a > 0$,

1) if f *is* ***even*** *:* $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$





2) if g *is* ***odd*** *:* $\int_{-a}^{a} g(x)\,dx = 0.$

3 INTEGRAL CALCULATION METHOD

3.1. Change of variable

Theorem:

If G is a primitive of g $(G' = g)$ *and* f *a differentiable function; then we* ***can calculate*** *the integral* $\int g[f(x)]\; f'(x)\,dx$ *by setting* $u = f(x)$.

We obtain $\frac{du}{dx} = f'(x)$ *so that* $f'(x)\,dx = du$ *then*

$$\int g[f(x)]\; f'(x)\,dx = \int g[u]\,du = G[u] + c = G[f(x)] + c$$

Applications:

$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c \text{ pour } n \neq -1 \qquad \int e^{f(x)} f'(x)\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln(|f(x)|) + c \qquad \int a^{f(x)} f'(x)\,dx = (\log_a e)\,a^{f(x)} + c$$

$$\int \sin(f(x)) f'(x)\,dx = -\cos(f(x)) + c \qquad \int \frac{f'(x)}{1+(f(x))^2}\,dx = \arctan(f(x)) + c$$

$$\int \cos(f(x)) f'(x)\,dx = \sin(f(x)) + c \qquad \int \cosh(f(x)) f'(x)\,dx = \sinh(f(x)) + c$$

$$\int \frac{f'(x)}{\cos^2(f(x))}\,dx = \tan(f(x)) + c \qquad \int \sinh(f(x)) f'(x)\,dx = \cosh(f(x)) + c$$

$$\int \frac{f'(x)}{\sqrt{1-(f(x))^2}}\,dx = \arcsin(f(x)) + c = - \qquad \int \frac{f'(x)}{\cosh^2(f(x))}\,dx = \tanh(f(x)) + c$$

Exercise : *Calculate primitives of:*

1) $g : x \to g(x) = \frac{\sin x}{1+\cos^2 x}$ **2)** $h : x \to f(x) = \frac{1}{x\sqrt{1-\ln^2 x}}$

Correction

1) the function f is continuous on $\mathbb{R}$ (*quotient of continuous functions, denominator don't vanishes*) thus admits a primitive $F(x) = \int^x \frac{\sin t}{1+\cos^2 t} dt$ on $\mathbb{R}$.

Put $u(t) = \cos t$ then $du = u'(t)\,dt = \sin t\,dt$, we deduce

$$F(x) = \int^x \frac{\sin t}{1+\cos^2 t} dt = \int^{t=x} \frac{1}{1+u^2}\,du = \arctan u\Big|^{t=x} = \arctan(\cos t)\Big|^{t=x} = \arctan(\cos x) + C.$$

2) $x \in D_f \iff x \neq 0 \,and\, 1-\ln^2 x > 0 \iff x \neq 0 \,and\, \ln|x| < 1/2 \iff x \neq 0 \,and\, |x| < e^{1/2}$.
the function f is continuous on $]-e^{1/2}, 0[\cup]0, +e^{1/2}[$ (*composed of continuous functions, denominator don't vanishes*) thus admits a primitive $F(x) = \int^x \frac{1}{t\sqrt{1-\ln^2 t}} dt$ on $]-e^{1/2}, 0[\cup]0, +e^{1/2}[$.

Put $u(t) = \ln t$ then $du = u'(t)\,dt = \frac{1}{t}\,dt$, we deduce

$$F(x) = \int^x \frac{1}{t\sqrt{1-\ln^2 t}} dt = \int^{t=x} \frac{1}{\sqrt{1-u^2}}\,du = \arcsin u\Big|^{t=x} = \arcsin(\cos t)\Big|^{t=x} = \arcsin(\cos x) + C$$

Exercise

Look for the *Primitives of*

1) $f \;:\; x \to \frac{x^2+1}{x^3+3x}$ ***2)*** $f \;:\; x \to \frac{x}{\sqrt{1+x^2}}$.

Correction

1) f is continuous on $\mathbb{R}$ as a quotient of continuous functions, hence f admits a primitive on $\mathbb{R}$.

For $u = x^3 + 3x$ we get $f(x) = \frac{u'}{3\,u}$ which is a derivative of $\frac{1}{3}\ln|u|$ (***see table of primitives***).

Therefore, primitives of f are the functions

$$F\ :\ x \to \tfrac{1}{3}\ln|x^3 + 3x| + c \qquad \text{where} \qquad c \in \mathbb{R}$$

2) / is continuous on $\mathbb{R}$ as composition and quotient of continuous functions, hence f admits a primitive on $\mathbb{R}$.

For $u = 1 + x^2$we have $f(x) = \frac{u'}{2\sqrt{u}}$ which is a derivative of $\sqrt{u}$ (*see table of primitives*).

Therefore, primitives of f are

$$F\ :\ x \to \sqrt{1 + x^2} + c \qquad\qquad \text{where} \quad c \in \mathbb{R}$$

Exercise

Calculate: **1)** $\int_{-2}^{0} \frac{1}{t^2+2t-3}\, dt$ **2)**$\int_{0}^{1} \frac{1}{x^2+x+1}\, dx$

Correction

1) $f(t) := \dfrac{1}{t^2 + 2t - 3}$ *in the denominator* $\Delta = 16$*, roots are* $t_1 = 1$ *and* $t_2 = -3$.

The denominator does not vanish on $(-2, 0)$*;* f is continuous and therefore integrable on $(-2, 0)$.

Let's decompose into simple element

$$\frac{1}{t^2 + 2t - 3} = \frac{1}{(t-1)(t+3)} = \frac{A}{t-1} + \frac{B}{t+3}$$

then multiplying by $(t - 1)$ and setting $t = 1$ we get $A = \frac{1}{(t+3)}\Big|_{t=1} = \frac{1}{4}$

then multiplying by we get $(t + 3)$ $B = \frac{1}{(t-1)}\Big|_{t=-3} = -\frac{1}{4}$

Consequently

$$\begin{aligned}\int_{-2}^{0} \frac{1}{t^2 + 2t - 3}\, dt &= \frac{1}{4}\int_{-2}^{0} \frac{1}{t-1}\, dt - \frac{1}{4}\int_{-2}^{0} \frac{1}{t+3}\, dt \\ &= \frac{1}{4}\ln|t-1|\ \Big|_{-2}^{0} - \frac{1}{4}\ln|t+3|\ \Big|_{-2}^{0} \\ &= -\frac{1}{4}\ln 3\ \ - \frac{1}{4}\ln 3 = -\frac{1}{2}\ln 3\end{aligned}$$

2) $f(x) := \frac{1}{x^2+x+1}$ ***in the denominator*** $\Delta < 0$, ***the denominator never vanish;*** f is continuous on $[0,1]$ and then f is integrable on $(0,1)$.

We put the trinomial in canonical form:

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1\right].$$

We make the change of variable $u = \dfrac{2x+1}{\sqrt{3}}$, we deduce $du = \frac{2}{\sqrt{3}}\,dx$; then

$$\int \frac{1}{x^2+x+1}\,dx = \frac{4}{3}\int \frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\,dx = \frac{4}{3}\int \frac{1}{u^2+1}\,\frac{\sqrt{3}}{2}\,du = \frac{4}{3}\,\frac{\sqrt{3}}{2}\arctan u + c$$

We have $x=0 \iff u = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ and $x=1 \iff u = \dfrac{3}{\sqrt{3}} = \sqrt{3}$. We conclude

$$\int_0^1 \frac{1}{x^2+x+1}\,dx = \frac{2}{\sqrt{3}}\arctan u\Big|_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} = \frac{2}{\sqrt{3}}\left(\arctan\sqrt{3} - \arctan\frac{\sqrt{3}}{3}\right) = \frac{2\sqrt{3}}{3}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi\sqrt{3}}{9}.$$

Exercise

Calculate primitives of: **1)** $f : x \to f(x) = e^x\cos(e^x)$ **2)** $f : x \longmapsto \dfrac{x^2+x-1}{x^4-1}$.

Correction

1) the function f is continuous on $\mathbb{R}$ (*product of continuous functions*) thus admits a primitive $F(x) = \int^x e^t \cos e^t dt$ on $\mathbb{R}$.

Put $u(t) = e^t$ then $du = u'(t)\,dt = e^t\,dt$, we deduce

$$F(x) = \int^x e^t \cos e^t dt = \int^{t=x} \cos u\,du = \sin u\Big|^{t=x} = \sin e^t\Big|^{t=x} = \sin e^x + C$$

2) the function f is continuous on $\mathbb{R}\backslash\{-1;1\}$ thus admits a primitive on $I_1 =]-\infty,-1[$, $I_2 =]-1,1[$ and $I_3 =]1,+\infty[$. Since $x^4-1 = (x-1)(x+1)(x^2+1)$ we have to seek real numbers a, b, c, d such that

for all $x \in \mathbb{R}\setminus\{-1,+1\}$: $$f(x) = \frac{a}{x-1} + \frac{b}{x+1} + \frac{cx+d}{x^2+1}.$$

$$a = \lim_{x\to 1}(x-1)\,f(x) = \frac{1}{4}, \quad b = \lim_{x\to -1}(x+1)\,f(x) = \frac{1}{4} \text{ and}$$

$$c\,i + d = \lim_{x\to i}(x^2+1)\,f(x) = 1 - \frac{i}{2} \implies c = -\frac{1}{2} \;et\; d = 1;$$

then $f(x)=\frac{1}{4}\left(\frac{1}{x-1}+\frac{1}{x+1}\right)+\frac{-\frac{x}{2}+1}{x^2+1}=\frac{1}{4}\left(\frac{1}{x-1}+\frac{1}{x+1}\right)-\frac{1}{2}\left(\frac{x}{x^2+1}\right)+\frac{1}{x^2+1}.$

We deduce primitives of f :

$$F(x)=\frac{1}{4}\Big(\ln|x-1|+\ln|x+1|\Big)-\frac{1}{4}\ln|x^2+1|+\arctan x+C$$

3.2. Integration by parts

Theorem :

Let u and v be two functions of class C^1 on an interval I, and a and b be two reals of I;. then

$$\int_a^b u(x)v'(x).dx=[u(x)v(x)]_a^b-\int_a^b u'(x)v(x).dx.$$

NB: This technique applies well to products of **polynomials with trigonometric** functions ($\cos x$, $\sin x$) or **exponential** functions ($\exp x$).

Exercise

1) Calculate the antiderivative $x\to x\sin x$ that vanishes at 0.

2) Calculate: $\int x\,e^{-x}\ dx$

Correction

1) *We are looking for the function $F(x)=\int_0^x t\,sint\ dt$. The function $x\to x\sin x$ is continuous on $\mathbb{R}$ and therefore is integrable. Let's use the by parts rule*

Put $u(t)=t$ and $v'(t)=\sin t$, then $u'(t)=1$ and $v(t)=-\cos t$.

For any real x, we have:

$$F(x)=\int_0^x t\,sint\ dt=\int_0^x u\,v'\ dt=-\ t\cos t\Big|_0^x+\int_0^x \cos t\ dt=-\ x\cos x+\sin t\Big|_0^x$$

Consequently $F(x)=-\ x\cos x+\sin x.$

2) $x\to xe^{-x}$ is continuous on $\mathbb{R}$, so integrable. Let's integrate by parts.

Put $u(x)=x$ and $v'(x)=e^{-x}$, then $u'(x)=1$ and $v(x)=-e^{-x}$.

$$\int x\,e^{-x}\ dx=\int u\,v'\ dx=-\,xe^{-x}+\int e^{-x}\ dx=-\,xe^{-x}-e^{-x}+c \qquad c\in\mathbb{R}$$

Exercise

Calculate 1) $\int \cos x\, e^{\lambda x}\ dx$ 2) $\int \sin t\, e^{\lambda t}\ dt$

Correction

1) The function $x \to \cos x\, e^{\lambda x}$ *(product of continuous functions) is continuous on* $\mathbb{R}$ *and therefore is integrable. Let us Integrate by parts :*

Put $u(x) = \cos x$ *and* $v'(x) = e^{\lambda x}$, *then* $u'(x) = -\sin x$ *and* $v(x) = \frac{1}{\lambda} e^{\lambda x}$.

Then

$$\int \cos x\, e^{\lambda x}\ dx = \int u\, v'\ dx = \frac{1}{\lambda} \cos x\, e^{\lambda x} + \frac{1}{\lambda} \int \sin x\, e^{\lambda x}\ dx \qquad \text{(EQ)}$$

To calculate $\int \sin x\, e^{\lambda x}\ dx$ let us *set* $u = \sin x$ *and* $v' = e^{\lambda x}$, *then* $u' = \cos x$ *and* $v(x) = \frac{1}{\lambda} e^{\lambda x}$.

$$\int \sin x\, e^{\lambda x}\ dx = \int u\, v'\ dx = \frac{1}{\lambda} \sin x\, e^{\lambda x} - \frac{1}{\lambda} \int \cos x\, e^{\lambda x}\ dx$$

We replace in (Eq)

$$\int \cos x\, e^{\lambda x}\, dx = \frac{1}{\lambda} \cos x\, e^{\lambda x} + \frac{1}{\lambda}\left[\frac{1}{\lambda} \sin x\, e^{\lambda x} - \frac{1}{\lambda} \int \cos x\, e^{\lambda x}\, dx\right] = \frac{1}{\lambda} \cos x\, e^{\lambda x} + \frac{1}{\lambda^2} \sin x\, e^{\lambda x} - \frac{1}{\lambda^2} \int \cos x\, e^{\lambda x}\, dx.$$

Hence
$$\left(1 + \frac{1}{\lambda^2}\right) \int \cos x\, e^{\lambda x}\ dx = \frac{1}{\lambda} \cos x\, e^{\lambda x} + \frac{1}{\lambda^2} \sin x\, e^{\lambda x},$$

we deduce
$$\int \cos x\, e^{\lambda x}\ dx = \frac{\lambda}{\lambda^2 + 1} \cos x\, e^{\lambda x} + \frac{1}{\lambda^2 + 1} \sin x\, e^{\lambda x}$$

2) The function $t \to \sin t\, e^{\lambda t}$ *(product of continuous functions) is continuous on* $\mathbb{R}$, *therefore integrable. To integrate by parts :*

Put $u = \sin x$ *and* $v' = e^{\lambda x}$, *then* $u' = \cos x$ *and* $v(x) = \frac{1}{\lambda} e^{\lambda x}$.

$$\int \sin x\, e^{\lambda x}\ dx = \int u\, v'\ dx = \frac{1}{\lambda} \sin x\, e^{\lambda x} - \frac{1}{\lambda} \int \cos x\, e^{\lambda x}\ dx \qquad \text{(EQ)}$$

Let's calculate $\int \cos x\, e^{\lambda x}\ dx$.

Set $u(x) = \cos x$ *and* $v'(x) = e^{\lambda x}$, *then* $u'(x) = -\sin x$ *and* $v(x) = \frac{1}{\lambda} e^{\lambda x}$.

$$\int \cos x\, e^{\lambda x}\ dx = \int u\, v'\ dx = \frac{1}{\lambda} \cos x\, e^{\lambda x} + \frac{1}{\lambda} \int \sin x\, e^{\lambda x}\ dx$$

We replace in (Eq)

$$\int \sin x\, e^{\lambda x}\, dx = \frac{1}{\lambda} \sin x\, e^{\lambda x} - \frac{1}{\lambda}\left[\frac{1}{\lambda} \cos x\, e^{\lambda x} + \frac{1}{\lambda} \int \sin x\, e^{\lambda x}\, dx\right] = \frac{1}{\lambda} \sin x\, e^{\lambda x} - \frac{1}{\lambda^2} \cos x\, e^{\lambda x} - \frac{1}{\lambda^2} \int \sin x\, e^{\lambda x}\, dx].$$

Hence $$\left(1+\frac{1}{\lambda^2}\right)\int \sin x\, e^{\lambda x}\, dx = \frac{1}{\lambda}\sin x\, e^{\lambda x} - \frac{1}{\lambda^2}\cos x\, e^{\lambda x},$$

we deduce $$\int \sin x\, e^{\lambda x}\, dx = \frac{\lambda}{\lambda^2+1}\sin x\, e^{\lambda x} - \frac{1}{\lambda^2+1}\cos x\, e^{\lambda x}$$

3.3. Integration of rational functions

Let's see in an example how to integrate the rational function

$$f(x) = \frac{\alpha x+\beta}{ax^2+bx+c}$$

with $\alpha, \beta, a, b, c \in \mathbb{R}$, $a \neq 0$ and $(\alpha,\beta) \neq (0,0)$

Consider the function $f(x) = \dfrac{x+1}{2x^2+x+1}$. First, we try to write a fraction of type $\dfrac{u'}{u}$ (*which we know how to integrate in* $\ln|u|$). We have $u = 2x^2+x+1$ then $u' = 4x+1$.

$$\begin{aligned} f(x) &= \frac{x+1}{2x^2+x+1} = \frac{\frac{1}{4}4x+\frac{1}{4}-\frac{1}{4}+1}{2x^2+x+1} \\ &= \frac{\frac{1}{4}(4x+1)\ +\ 3/4}{2x^2+x+1} = \frac{1}{4}\frac{4x+1}{2x^2+x+1} + \frac{3}{4}\frac{1}{2x^2+x+1} \\ &= E(x)\ +\ D(x). \end{aligned}$$

For the **first part E(x)** we have

$$\int E(x)dx = \frac{1}{4}\int \frac{4x+1}{2x^2+x+1}\,dx = \frac{1}{4}\int \frac{u'}{u}\,dx = \frac{1}{4}\ln|u| + Cst = \frac{1}{4}\ln|2x^2+x+1| + Cst.$$

For the **second part D(x)**, three basic situations ***can occur in general***:

First situation: The denominator $ax^2 = bx+c$ has two distinct real roots $x_1, x_2 \in \mathbb{R}$. Then $f(x)$ can be written

$$f(x) = \frac{\alpha x+\beta}{a(x-x_1)(x-x_2)} = \frac{A}{x-x_1} + \frac{B}{x-x_2}.$$

Integrating, we'll get we have

$$\int f(x)dx = A\,\ln|x-x_1| + B\,\ln|x-x_2| + Cst.$$

on each of the intervals $]-\infty, x_1[,\ ]x_1, x_2[,\ ]x_2, +\infty[$ $(x1 < x2)$.

Second situation: The denominator $ax^2 = bx + c$ has a double root $x_0 \in \mathbb{R}$. Then $f(x)$ can be written

$$f(x) = \frac{\alpha x + \beta}{a(x-x_0)^2} = \frac{A}{(x-x_0)^2} + \frac{B}{x-x_0}.$$

Integrating, we'll get we have

$$\int f(x)dx = -\frac{A}{x-x_0} + B\,\ln|x-x_0| + Cst.$$

on each of the intervals $]-\infty, x_0[,\ \ ]x_0, +\infty[$.

Third situation: The denominator $ax^2 = bx + c$ has no real root. We will see this case later.

Let us return to example below and consider the second part and let's calculate

$$\int D(x)dx = \int \frac{3}{4}\frac{1}{2x^2+x+1}dx.$$

In the **denominator** we have $\Delta = -7 < 0$, denominator has no real root, so we **will write it** in the form $u^2 + 1$ (which a primitive is **arctan(u)**)

$$\begin{aligned} Denominator &= 2x^2 + x + 1 = 2\left(x^2 + 2\,\tfrac{1}{4}\,x + \tfrac{1}{2}\right) \\ &= 2\left((x+\tfrac{1}{4})^2 - (\tfrac{1}{4})^2 + \tfrac{1}{2}\right) = 2\left((x+\tfrac{1}{4})^2 + \tfrac{7}{16}\right), \end{aligned}$$

then

$$\begin{aligned} \int D(x)dx &= \int \frac{3}{4}\ \frac{1}{2x^2+x+1}d\,x = \frac{3}{8}\ \int \frac{1}{(x+\frac{1}{4})^2 + \frac{7}{16}}\,dx \\ &= \frac{3}{8}\ \frac{16}{7}\ \int \frac{1}{[\frac{4}{\sqrt{7}}\,(x+\frac{1}{4})]^2 + 1}\,dx. \end{aligned}$$

Put $u = \left[\frac{4}{\sqrt{7}}\,(x+\frac{1}{4})\right]$ then, $du = \frac{4}{\sqrt{7}}\,dx$ i.e. $dx = \frac{\sqrt{7}}{4}\,du$ and we deduce

$$\begin{aligned} \int D(x)dx &= \frac{3}{8}\ \frac{16}{7}\ \int \frac{1}{[\frac{4}{\sqrt{7}}\,(x+\frac{1}{4})]^2+1}\,dx = \frac{3}{8}\times\frac{16}{7}\ \int \frac{1}{u^2+1}\,dx \times \frac{\sqrt{7}}{4} \\ &= \frac{3\sqrt{7}}{14}\ \arctan u + Cst = \frac{3\sqrt{7}}{14}\ \arctan\left[\tfrac{4}{\sqrt{7}}\,(x+\tfrac{1}{4})\right] + Cst. \end{aligned}$$

3.4. Integration of irreducible rational functions

Let $f(x) = \dfrac{P(x)}{Q(x)}$ a rational fraction, where P(x) and Q(x) are polynomials with real coefficients and $degre\, Q \leq 2$. So, the fraction $\dfrac{P(x)}{Q(x)}$ can be written as the sum of a polynomial E(x) (integer part) and simple elements D(x) of one of the following forms:

$$\frac{\gamma}{(x-x_0)^k} \qquad \text{or} \qquad \frac{\alpha x+\beta}{(ax^2+bx+c)^k} \text{ with } \Delta = b^2-4ac<0$$

where $\alpha, \beta, \gamma, a, b, c \in \mathbb{R}$, $k \in \mathbb{N}^*$, $a \neq 0$ and $(\alpha, \beta) \neq (0,0)$.

We can easily integrate the integer part E(x). Our interest is how to integrate the simple element part D(x).

Integration of $\dfrac{\gamma}{(x-x_0)^k}$:

1) If $k = 1$ then

$$\int \frac{\gamma}{x-x_0}\,dx = \gamma \ln|x-x_0| + Cst,$$

on each of the intervals $]-\infty, x_0[,\ ]x_0, +\infty[$.

2) If $k \geq 2$ then

$$\int \frac{\gamma}{(x-x_0)^k}\,dx = \gamma \int (x-x_0)^{-k}\,dx = \frac{\gamma}{-k+1}(x-x_0)^{-k+1} + Cst,$$

on each of the intervals $]-\infty, x_0[,\ ]x_0, +\infty[$.

Integration of $\dfrac{\alpha x+\beta}{(ax^2+bx+c)^k}$ with $\Delta = b^2-4ac<0$. First, we try to write a fraction of type $\dfrac{u'}{u^k}$ where $u = ax^2+bx+c$ then $u' = 2ax+b$:

$$\frac{\alpha x+\beta}{(ax^2+bx+c)^k} = \gamma\,\frac{2ax+b}{(ax^2+bx+c)^k} + \delta\,\frac{1}{(ax^2+bx+c)^k}$$

For the **first part** we have

1) If $k=1$ then

$$\int \gamma\ \frac{2ax+b}{ax^2+bx+c}dx = \gamma\ \int \frac{u'}{u}\,dx = \gamma\ \ln|u| + Cst = \gamma\ \ln|ax^2+bx+c| + Cst.$$

2) If $k \geq 2$ then

$$\begin{aligned}\int \gamma\ \frac{2ax+b}{(ax^2+bx+c)^k}dx &= \gamma\ \int \frac{u'}{u^k}\,dx = \gamma\ \int u^{-k}\ u'\,dx = \frac{\gamma}{-k+1}u^{-k+1} + Cst \\ &= \frac{\gamma}{-k+1}(ax^2+bx+c)^{-k+1} + Cst\end{aligned}.$$

For the **second part**; since $\Delta = b^2-4ac<0$ we have to write the denominator in the form $ax^2+bx+c = C\left(u^2+1\right)$ where u is in the form $u = px+q$, $du = p\,dx$.

1) If $k=1$ then

$$\int \delta\ \frac{1}{ax^2+bx+c}\,dx = \frac{\delta}{p\,C}\ \int \frac{1}{u^2+1}\,du = \frac{\delta}{p\,C}\arctan(px+q) + Cst.$$

2) If $k \geq 2$ then

$$\int \gamma\ \frac{1}{(ax^2+bx+c)^k}\,dx = \frac{\gamma}{p\,C}\ \int \frac{1}{(1+u^2)^k}\,du := \frac{\gamma}{p\,C}\ I_k.$$

An integration by part permits to pass to I_{k-1} …

Consider the function $f(x) = \dfrac{x+1}{2x^2+x+1}$. First, we try to write a fraction of type $\dfrac{u'}{u}$ (*which we know how to integrate in* $\ln|u|$). We have $u = 2x^2+x+1$ then $u' = 4x+1$.

3.5. Integration of trigonometric functions

To calculate primitives of the form $\int P(\cos x, \sin x)\,dx$ or of the form $\int \frac{P(\cos x,\sin x)}{Q(\cos x,\sin x)}\,dx$, where P and Q are polynomials, we can reduce calculus to integrating a rational fraction, using here following methods:

- ***Bioche's rules*** which are quite effective but do not always work;

- the ***change of variable*** $t = \tan(\frac{x}{2})$ works all the time but leads to more calculations.

The rules of Bioche: We note $w(x) = f(x)\,dx$, then we have $w(-x) = -f(-x)\,dx$ and $w(\pi - x) = -f(\pi - x)dx$.

- · If $w(-x) = w(x)$ then we perform the change of variable $u = \cos x$.
- · If $w(\pi - x) = w(x)$ then we perform the change of variable $u = \sin x$.
- · If $w(\pi + x) = w(x)$ then we perform the change of variable $u = \tan x$.

Exercise

Calculate $\displaystyle\int \frac{\cos x}{2 - \cos^2 x}\,dx$

Correction

$w(x) = \dfrac{\cos x}{2 - \cos^2 x}\,dx$, *we can verify that*

$$w(\pi - x) = \frac{\cos(\pi - x)}{2 - \cos^2(\pi - x)}\,d(\pi - x) = \frac{-\cos x}{2 - \cos^2 x}(-dx) = w(x).$$

So, we will we perform the change of variable $u = \sin x$, $du = \cos x$. *Then*

$$\int \frac{\cos x\,dx}{2 - \cos^2 x} = \int \frac{du}{2 - (1 - u^2)} = \int \frac{du}{1 + u^2} = \arctan u + Cst = \arctan(\sin x) + Cst.$$

The change of variable $t = \tan(\frac{x}{2})$: We use the rules.

$$\cos x = \frac{1 - t^2}{1 + t^2} \qquad \sin x = \frac{2t}{1 + t^2} \qquad \tan x = \frac{2t}{1 - t^2} \quad \text{and} \quad dx = \frac{2\,dt}{1 + t^2}.$$

Exercise

Calculate $\displaystyle\int_{-\pi/2}^{0} \frac{1}{1 - \sin x}\,dx$

Correction

Put $t = \tan(\frac{x}{2})$ *then* $\sin x = \dfrac{2t}{1 + t^2}$ *and* $dx = \dfrac{2\,dt}{1 + t^2}$, *consequently*

$$\int \frac{1}{1 - \sin x}\,dx = \int \frac{1}{1 - \frac{2t}{1+t^2}} \frac{2\,dt}{1 + t^2} = 2\int \frac{1}{1 + t^2 - 2t}\,dt = 2\int \frac{1}{(1 - t)^2}\,dt = 2\,\frac{1}{1 - t} + Cst.$$

$x = -\pi/2 \iff 2t = -1 - t^2 \iff (1+t)^2 = 0 \iff t = -1$ and $\quad x = 0 \iff t = 0$. Then

$$\int_{-\pi/2}^{0} \frac{1}{1-\sin x}\, dx = 2\, \frac{1}{1-t}\Big|_{-1}^{0} = 2\left(1 - \frac{1}{2}\right) = 1.$$

4 IMPROPER INTEGRALS

4.1. Definitions and examples

Definition :

- *We can extend the definition of the integral over $[a, b]$ with $a, b \in \mathbb{R}$:*

1) *to unbounded functions on an open interval.*

2) *to unbounded intervals*

- *The principle is to consider $\int_x^y f(t)\, dt$ then to **pass to the limit** $x \to a$ and/or $y \to b$.*

1) *If the **limit exists**, we talk about **convergent integral***

2) *If the **limit does not exist**, we deal with **divergent integral.***

- *This kind of integral is called "**improper integral**" or "**generalized integral**" .*

Example :

Riemann integral :	$\displaystyle\int_1^{+\infty} \frac{1}{t^\alpha}\, dt$	exists if and only if $\alpha > 1$.

Indeed,

- if $\alpha = 1$: $\displaystyle\int_1^{+\infty} \frac{1}{t}\, dt = \lim_{x\to+\infty} \int_1^{x} t\, dt = \lim_{x\to+\infty} \ln(x) = +\infty.$

  The integral diverges.

- If $\alpha \neq 1$

  $$\int_1^{+\infty} \frac{1}{t^\alpha} = \lim_{x\to+\infty} \int_1^{x} t^{-\alpha}\, dt = \lim_{x\to+\infty} \frac{t^{-\alpha+1}}{-\alpha+1}\Big|_1^{x} = \frac{1}{-\alpha+1}\Big[\lim_{x\to+\infty} x^{-\alpha+1} - 1\Big].$$

- For $\alpha > 1$:
$$\int_1^{+\infty} \frac{1}{t^\alpha} = \lim_{x\to+\infty} \int_1^x t^{-\alpha}\, dt = \frac{1}{-\alpha+1}\Big[\lim_{x\to+\infty} x^{-\alpha+1} - 1\Big] = -\frac{1}{-\alpha+1}.$$
The integral converges.
- For $\alpha < 1$:
$$\int_1^{+\infty} \frac{1}{t^\alpha} = \lim_{x\to+\infty} \int_1^x t^{-\alpha}\, dt = \frac{1}{-\alpha+1}\Big[\lim_{x\to+\infty} x^{-\alpha+1} - 1\Big] = +\infty.$$
The integral diverges.

Exercise 1 :

Study the existence of integrals ***1)*** $\int_0^1 \frac{1}{\sqrt{t}}\, dt$ ***2)*** $\int_0^1 \frac{1}{t}\, dt.$

Correction

1) *The function* $t \to \frac{1}{\sqrt{t}}$ *is continuous on* $]0,1]$ *but not bounded at 0.*

For all $0 < x < 1$ *the function can be integrated on* $]x,1]$ *and*

$$I(x) := \int_x^1 \frac{1}{\sqrt{t}} = 2\sqrt{t}\Big|_x^1 = 2 - 2\sqrt{x}$$

Passing to the limit : $\lim_{x\to 0^+} I(x) = \lim_{x\to 0^+} 2 - 2\sqrt{x} = 2.$

$\int_0^1 \frac{1}{\sqrt{t}}\, dt$ is a generalized integral that exists. We have $\int_0^1 \frac{1}{\sqrt{t}}\, dt = 2.$

2). *The function* $t \to \frac{1}{t}$ *is continuous on* $]0,1]$ *but not bounded at 0.*

For all $0 < x < 1$ *the function can be integrated on* $]x,1]$ *and :*

$$I(x) := \int_x^1 \frac{1}{t} = \ln(t)\Big|_x^1 = \ln 1 - \ln x = -\ln x$$

We pass to the limit: $\lim_{x\to 0^+} I(x) = \lim_{x\to 0^+} -\ln x = +\infty.$

$\int_0^1 \frac{1}{t}\, dt$ is a generalized integral which does not exist.

The generalized integral $\int_0^1 \frac{1}{t}\, dt$. **diverges.**

Exercise 2:

Study the existence of integrals ***1)*** $\int_0^\infty e^{-t}\, dt$ ***2)*** $\int_{-\infty}^{+\infty} \frac{1}{1+t^2}\, dt.$

Correction

1) *The function $t \to e^{-t}$ is continuous on $[0,+\infty[$ but the interval is unbounded.*

For all $x > 0$ the function is integrable on $[0,x]$ and :

$$I(x) := \int_0^x e^{-t}\, dt = -e^{-t}\Big|_0^x = -e^{-x} + 1$$

We pass to the limit: $\lim\limits_{x\to+\infty} I(x) = \lim\limits_{x\to+\infty} -e^{-x} + 1 = +1.$

$\int_0^\infty e^{-t}\, dt$ is a generalized integral that exists. We have $\int_0^\infty e^{-t}\, dt = 1.$

2) *The function $t \to \dfrac{1}{1+t^2}$ is continuous on $]-\infty,+\infty[$ but the interval is unbounded.*

For all $-\infty < x < y < +\infty$ the function can be integrated on $[x,y]$ and :

$$I(x,y) := \int_x^y \frac{1}{1+t^2}\, dt = \arctan t\Big|_x^y = \arctan y - \arctan x.$$

Let's pass to the limit: $\lim\limits_{y\to+\infty} \arctan y = \dfrac{\pi}{2}$ and $\lim\limits_{x\to-\infty} \arctan x = -\dfrac{\pi}{2}.$

Consequently $\int_{-\infty}^{+\infty} \dfrac{1}{1+t^2}\, dt$ is a generalized integral that exists.

We have $\int_{-\infty}^{+\infty} \dfrac{1}{1+t^2}\, dt = \dfrac{\pi}{2} - (-\dfrac{\pi}{2}) = \pi.$

4.2. Convergence criteria

We will note $\int_a^{\to b} \dfrac{1}{1+t^2}\, dt := \lim_{y\to b} \int_a^y \dfrac{1}{1+t^2}\, dt$ and $\int_{\to a}^{b} \dfrac{1}{1+t^2}\, dt := \lim_{x\to b} \int_x^a \dfrac{1}{1+t^2}\, dt$.

> ***Theorem (comparison criterions):***
>
> *Let f and g be two functions defined on an interval $[a,b[\subset \mathbb{R}$.*
>
> *Under the assumptions that $\forall t \in [a,b[\ :\ 0 \le f(t) \le g(t)$,*
>
> ***1)*** *if $\int_a^{\to b} g(t)\, dt$ converges then $\int_a^{\to b} f(t)\, dt$ converges and*
>
> $$\int_a^{\to b} f(t)\, dt \le \int_a^{\to b} g(t)\, dt;$$
>
> ***2)*** *if $\int_a^{\to b} f(t)\, dt$ diverges then $\int_a^{\to b} g(t)\, dt$ diverges.*

Exercise:

Study the convergence of the integral $$\int_0^{+\infty} \frac{\sin^2 t}{1+t^2}\, dt.$$

Correction

1). $\forall t \in [0,+\infty[\,:\ 0 \leq \dfrac{\sin^2 t}{1+t^2} \leq \dfrac{1}{1+t^2}$ **and**

$$\int_0^{+\infty} \frac{1}{1+t^2}\, dt = \lim_{x\to+\infty} \int_0^x \frac{1}{1+t^2}\, dt = \lim_{x\to+\infty} \arctan x = \frac{\pi}{2}.$$

We deduce that $\int_0^{+\infty} \dfrac{\sin^2 t}{1+t^2}\, dt$ **converges** **and we have** $0 \leq \int_0^{+\infty} \dfrac{\sin^2 t}{1+t^2}\, dt \leq \dfrac{\pi}{2}.$

> ***Theorem (equivalence criterions):***
>
> *Let f and g, be two functions defined on an interval $[a,b[\subset \mathbb{R}$;*
>
> *we assume that: $f \underset{b}{\sim} g$ (f and g are **equivalent** close to b) then*
>
> $\int_a^{\to b} g(t)\, dt$ *and* $\int_a^{\to b} f(t)\, dt$ *are of **similarly nature**.* ***(both converge or both diverge)***

Exercise:

Determine the nature of the integral $$\int_{\to 0}^{1} \frac{t^\alpha}{[\ln(1+t)]^\beta}\, dt.$$

Correction

$\ln(1+t) \underset{0}{\sim} t \Longrightarrow \frac{t^\alpha}{[\ln(1+t)]^\beta} \underset{0}{\sim} t^{\alpha-\beta}$ **and**

$$\int_{\to 0}^{1} t^{\alpha-\beta}\, dt = \lim_{x\to 0} \int_x^1 t^{\alpha-\beta}\, dt = \frac{1}{\alpha-\beta+1} \lim_{x\to 0} t^{\alpha-\beta+1}\Big|_x^1 = \frac{1}{\alpha-\beta+1} \lim_{x\to 0} \left[1 - x^{\alpha-\beta+1}\right]$$

We deduce that $\int_{\to 0}^{1} \frac{t^\alpha}{[\ln(1+t)]^\beta}\, dt$ **converges if and only if** $\alpha-\beta+1>0$ **i.e.** $\beta-\alpha<1$.

Example: **(**Euler's Gamma function**)**

> ***Euler's Gamma function*** *is defined by :*
>
> $$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt \quad : \ x>0.$$
>
> It is a convergent generalized integral.

For $0 < t < 1$ we have $e^{-t} < 1$ and then $\forall x \in \mathbb{R}$:

$$\int_0^1 t^{x-1} e^{-t}\, dt \leq \int_0^1 t^{x-1}\, dt = \frac{1}{x}.$$

For $t \geq 1$ we have $\forall x \in \mathbb{R}$, $t^{x-1} \leq t^x$ and $\max_{t\geq 1} t^x e^{-t/2} = (2x)^x e^{-x}$; consequently

$$\int_1^{\infty} t^{x-1} e^{-t}\, dt \leq \int_1^{\infty} t^x e^{-t/2}\, e^{-t/2}\, dt \leq (2x)^x e^{-x} \int_1^{\infty} e^{-t/2}\, dt = 2\,[(2x)^x e^{-x} - e^{-1/2}] < 2\,(2x)^x e^{-x}.$$

Properties :

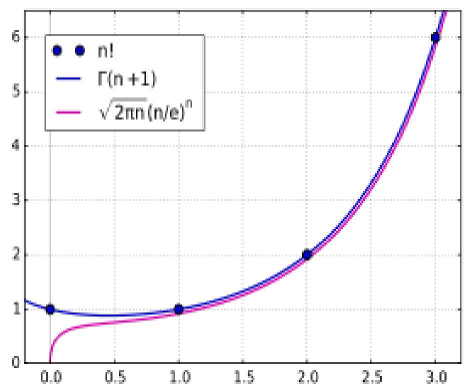
1) *For every* $n \in \mathbb{Z}_+$ *:* $\Gamma(n+1) = n!$,

2) *for every* $x > 0$ *:* $\Gamma(x+1) = x\,\Gamma(x)$,

3) $\Gamma(1/2) = \sqrt{\pi}$

4) *Stirling's approximation:*

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}$$



Note: the formula $\Gamma(n+1) = n!$ shows that the gamma function is an ***extension of the notion of "factorial".***

4.3. Absolute convergence

Definition

*Let f be a function defined on $[a, b[\subset \mathbb{R}$; we say that f is **absolutely integrable** or that $\int_a^{\to b} f(t)\, dt$ is **absolutely convergent** if*

$$\int_a^{\to b} |f(t)|\, dt < \infty.$$

Theorem:

absolutely convergent integral $\Longrightarrow$ convergent integral.

Example:

Let us study the convergence of the integral $\displaystyle\int_1^{+\infty} \frac{\sin t}{t^\alpha}\, dt$ for $\alpha > 1$.

We have

$$\int_1^{+\infty} \left| \frac{\sin t}{t^\alpha} \right| dt \leq \int_1^{+\infty} \frac{1}{t^\alpha}\, dt < \infty$$

because $\int_1^{+\infty} \frac{1}{t^\alpha}\, dt$ is a Riemann-type integral which converges for $\alpha > 1$

So $\int_1^{+\infty} \frac{\sin t}{t^\alpha}\, dt$ is an **absolutely convergent integral** and from the above theorem we deduce that $\int_1^{+\infty} \frac{\sin t}{t^\alpha}\, dt$ is a **convergent**.

Chap 3 : Ordinary Differential Equations (ODE)

1 DEFINITIONS

Definition (differential equation ordinary):

- ✓ *An* **ordinary differential equation** *is a relation between an unknown function (of one variable x) and its derivatives.*

$$F(x, y, y', y'', ...) = 0.$$

- ✓ *The* **order** *of a differential equation corresponds to the* ***maximum degree of differentiation*** *to which the unknown function has been subjected.*
  Thus, a ***differential equation of order n*** *is an equation of the form*

$$F(x, y, y', y'', ..., y^{(n)}) = 0$$

- ✓ *If the coefficients relating to the unknowns $y, y'y'', ...$are constants, the* **equation is said to** *have* **constant coefficients**.
- ✓ *If the coefficients relating to the unknowns $y, y'y'', ...$are functions of x, the* **equation is said to** *have* **variable coefficients**.
- ✓ *Any function that satisfies, for all x, the equation $F(x, y, y', y'', ...) = 0$ is a* **solution** *of this equation.*
- ✓ **Solving** *(or* **integrating** *) a differential equation consists in* ***determining*** *the set of* ***functions*** *which are* ***solutions to it*** *.*

2 FIRST ORDER ODE

A differential equation of the **prime order** *is of the form*

$$F(x, y, y') = 0 \qquad \text{or} \qquad y' = f(x, y).$$

Examples.

1) Here are some easy to solve differential equations. Find a function, solution of the following differential equations:

a) $y' = sinx$ *b)* $y' = 1 + e^x$ c) $y' = y$

Answer : These are (simple) first-order differential equations. The **solutions** are the primitives of the second member:

a) $y = -\cos x + k \quad (k \in \mathbb{R})$ b) $y = x + e^x + k \quad (k \in \mathbb{R})$, c) $y = k\, e^x \quad (k \in \mathbb{R})$

2) Consider the differential equation $y' = 2x\, y + 4x$.
Check that $y(x) = k \exp(x2) - 2 \quad (k \in \mathbb{R})$ is a solution on $\mathbb{R}$.

Answer : These are first order differential equations with variable coefficients. We replace $y(x)$ to see if it satisfies the equation:

on the left: $y' = [k \exp(x2) - 2]' = 2k \exp(x2)$

on the right: $2x\, y + 4x = 2x\,[k \exp(x2) - 2] + 4x = 2x\, k \exp(x2) - 4x + 4x = 2x\, k \exp(x2)$

therefore effectively $y(x) = k \exp(x2) - 2 \quad (k \in \mathbb{R})$ is a solution of the differential equation $y' = 2x\, y + 4x$ for all $k \in \mathbb{R}$.

3) Same question for $x^2\, y'' - 2x\, y + 2x = 0$ and $y(x) = k\, x^2 + x \quad (k \in \mathbb{R})$.

Answer : These are second-order differential equations with variable coefficients. We substitute $y(x)$ to see if it satisfies the equation:

we have $y(x) = k\, x^2 + x$, $y' = 2k\, x + 1$ and $y'' = 2k$.

We replace in the equation
$x^2\, y'' - 2x\, y + 2x = x^2\,(2k) - 2x\,(2k\, x + 1) + 2x = 2k\, x^2 - 2x\,(2k\, x + 1) + 2x$=0,
therefore effectively $y(x) = k \exp(x2) - 2 \quad (k \in \mathbb{R})$ is a solution of the differential equation $y' = 2x\, y + 4x$ for all $k \in \mathbb{R}$.

2.1. Linear 1st order ODE

The linear case can be written more simply:

$$y' + p(x)\, y = q(x)$$

Multiplying by $\mu(x) > 0$, said **integration factor**, we get

$$\mu(x)\, y' + \mu(x)\, p(x)\, y = \mu(x)\, q(x).$$

We Look for $\mu(x) > 0$ such as $\mu'(x) = \mu(x)\, p(x)$:

$$\mu'(x) = \mu(x)\, p(x) \iff \mu(x) = e^{\int p(x) dx}$$

In this case we have $\mu(x)\,y' + \mu'(x)\,y = \big(\mu(x)\,y\big)' = \mu(x)\,q(x)$,

then we can deduce the solution

$$y = \frac{1}{\mu(x)} \int \mu(x)\,q(x)\ dx.$$

Method :

to solve the differential equation $y' + p(x)\,y = q(x)$

- ✓ *we calculation of the integrating factor* $\mu(x) = e^{\int p(x)dx}$*;*
- ✓ *by multiplying by* $\mu(x)$ *we obtain an equation of the type*

$$\mu(x)\,y' + \mu'(x)\,y = \big(\mu(x)\,y\big)' = \mu(x)\,q(x).$$

- ✓ *We easily integrate* $\big(\mu(x)\,y\big)' = \mu(x)\,q(x)$ *to find*

$$y = \frac{1}{\mu(x)} \int \mu(x)\,q(x)\ dx.$$

Exercise

Solve the following differential equations.

1) $y' + \frac{2}{x}\,y = 6x^3$ ***2)*** $\tan x\ \frac{dy}{dx} + y = \frac{1}{\cos x}$.

3) $y' - xy = x$. ***4)*** $t\,y' + 2y = 4t^2$. with $y(1) = 2$.

Correction

1) $y' + \frac{2}{x}\,y = 6x^3$ is a first order linear equation.

Its integrating factor is $\mu(x) = e^{\int \frac{2}{x}\,dx} = e^{\ln x^2} = x^2$.

Multiplying the equation by $\mu(x) = x^2$ we get:

$$x^2\,y' + 2x\,y = x^2\,y' + (x^2)'\,y = (x^2\,y)' = 6x^5;$$

hence by integrating we'll get

$$x^2 y = x^6 + c \qquad \textit{or} \qquad y = x^4 + \frac{c}{x^2}.$$

2) $\tan x\ \frac{dy}{dx} + y = \frac{1}{\cos x}$: it is a first order linear equation. let's write

$$\frac{dy}{dx} + \cot x\ y = \frac{1}{\sin x},$$

the integrating factor is given by $\qquad \mu(x) = e^{\int \frac{\cos x\, dx}{\sin x}} = e^{\int \frac{d(\sin x)}{\sin x}} = e^{\ln \sin x} = \sin x.$

Multiplying the equation by $\mu(x) = \sin x$ we'll get:

$$\sin x\ dy + (\cos x dx)\ y = d(y\ \sin x) = 1\ dx;$$

Integrating, we get the solution:

$$y\ \sin x = x + c.$$

3) $y' - xy = x$ is a first order linear equation.

Its integrating factor is $\quad \mu(x) = e^{\int -x\, dx} = e^{-\frac{x^2}{2}}.$

Multiplying the equation by $\mu(x) = e^{-\frac{x^2}{2}}$ we get:

$$e^{-\frac{x^2}{2}}\, y' + (-xe^{-\frac{x^2}{2}})\ y = e^{-\frac{x^2}{2}}\, y' + (e^{-\frac{x^2}{2}})'\ y = (e^{-\frac{x^2}{2}}\ y)' = xe^{-\frac{x^2}{2}};$$

By integrating we obtain:

$$y\, e^{-\frac{x^2}{2}} = -e^{-\frac{x^2}{2}} + c \qquad \textit{or} \qquad y = -1 + c\, e^{\frac{x^2}{2}}.$$

4) $t\, y' + 2y = 4t^2$. with initial condition $y(1) = 2$. it is a first order linear equation which is written

$$y' + \tfrac{2}{t}\, y = 4t$$

Its integrating factor is
$\mu(t) = e^{\int \frac{2}{t}\, dt} = e^{\ln t^2} = t^2.$

Multiplying the equation by $\mu(t) = t^2$ we get:

$$t^2\, y' + 2t\, y = (t^2\, y)' = t^4;$$

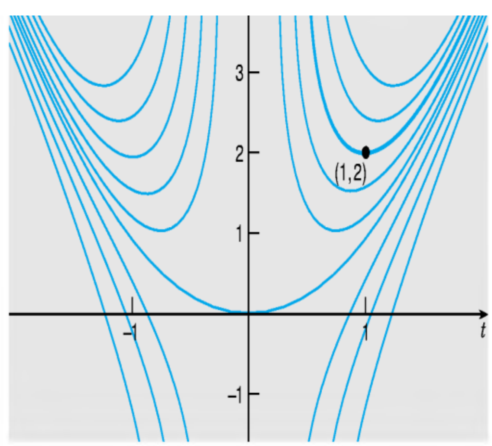
hence by integrating

$$t^2 y = t^4 + c \qquad \textit{or} \qquad y = t^2 + \tfrac{c}{t^2}.$$

With the initial condition $y(1) = 2$ we will have
$(1)^2\ 2 = (1)^4 + c \iff c = 1.$

Thus the particular solution is

$$y = t^2 + \tfrac{1}{t^2}.$$



2.2. 1st Order ODE Separable

A first-order ODE can be presented in the form

$$M(x,y)dx + N(x,y)dy = 0 \ \ -(*).$$

There is no general method but an interesting special case is when the functions are $M(x,y)\ et\ N(x,y)$ separable variables **i.e.** :

$$M(x,y) = M_1(x)\, M_2(y) \qquad \text{and} \qquad N(x,y) = N_1(x)\, N_2(y).$$

then the **equation** $(*)$is said **to have separable variables** and the solution is obtained by simple integration

> ***Method:***
> *with the hypothesis* $M_2(x) \neq 0$, $N_1(x) \neq 0$ *we can write:*
>
> $$(*) \iff \frac{M_1(x)}{N_1(x)}dx + \frac{N_2(y)}{M_2(y)}dy = 0,$$
>
> *then we integrate.*

Example :

Consider the differential equation $\qquad xdx + ydy = 0.$
It is an equation with separated variables. By integrating on both sides

$$\int xdx + \int ydy = c_1 \qquad \text{either} \qquad \frac{x^2}{2} + \frac{y^2}{2} = c_1$$

with $c_1 > 0$. Let $2c_1 = c^2$, then the ***general solution*** is given by

$$x^2 + y^2 = c^2$$

which represents a ***family of concentric circles*** centered at the origin of the coordinates and of radius c.

Exercise

Integrate the following differential equations.

1) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$; ***2)*** $\frac{dy}{dx} + \frac{1+y^3}{xy^2(1+x^2)} = 0$; ***3)*** $\frac{dy}{dx} = \frac{cos^2 y}{\sin^2 x}$.

Correction

1) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$ **can be written in the form** $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$.

By integrating in both sides we get: $\arctan y = \arctan x + c$, **or**

$$y = \tan(\arctan x + c).$$

NB: **if we pose** $c = \arctan k$ **then the trigonometric formula** $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \, \tan b}$ **will allow us to write**

$$y = \tan(\arctan x + (\arctan k) = \frac{x+k}{1-kx}.$$

2) $\frac{dy}{dx} + \frac{1+y^3}{xy^2(1+x^2)} = 0$ **can be written in the form**

$\frac{y^2}{1+y^3}dy = \frac{-1}{x(1+x^2)}dx = \frac{A}{x}dx + \frac{Bx+D}{1+x^2}dx.$

Multiplying by x**we have** $\frac{-1}{(1+x^2)} = A + \frac{(Bx+D)x}{1+x^2}$ **for** $x = 0$ **we get** $A = -1$.

Multiplying by $1 + x^2$**we have** $\frac{-1}{x} = \frac{A(1+x^2)}{x} + Bx + D$; **for** $x = i$ **we'll have** $i = Bi + D$ **then** $B = 1$ **and** $D = 0$.

The equation becomes

$$\frac{1}{3}\,\frac{3y^2}{1+y^3}dy = -\,\frac{1}{x}dx + \frac{1}{2}\,\frac{2x}{1+x^2}dx.$$

Integrating we obtain:

$$\tfrac{1}{3}\,\ln|1+y^3| = -\ln|x| + \tfrac{1}{2}\,\ln|1+x^2| + c,$$

multiplying by 6 and using the properties of the logarithm we'll have:

$$\ln\frac{(1+y^3)^2\,x^6}{(1+x^2)^3} = 6c \qquad \textit{or} \qquad \frac{x^6\,(1+y^3)^2}{(1+x^2)^3} = e^{6c} = c'.$$

3) $\frac{dy}{dx} = \frac{cos^2 y}{\sin^2 x}$. **Can be written** $\frac{1}{cos^2 y}dy = -\,\frac{-1}{\sin^2 x}dx$,

by integrating (see table of primitives) we obtain:

$$\tan y = -\cot x + c \qquad \text{or} \qquad y = \arctan(-\cot x \ + c).$$

3 SECOND-ORDER LINEAR EQUATION WITH CONSTANT COEFFICIENTS

- ✓ *A differential equation of the* **second order** *is of the form*
  $F(x, y, y', y'')$ *or* $y'' = f(x, y, y')$.
- ✓ *A* **linear** *differential equation of the* **second order** *is of the form*
  $$a\,y'' + b\,y' + c\,y = f(x).$$
- ✓ *A linear differential equation of the form*
  $$a\,y'' + b\,y' + c\,y = 0.$$
  is said to be **homogeneous** *or* **without second order**.

3.1. Homogeneous equation with constant coefficients

Consider the **linear** and **homogeneous** differential equation with **constant coefficients** of **order 2**:

$$(a \neq 0) \quad ay'' + by' + cy = 0 \quad --E_h$$

where a, b and c are real constants.

Let's look for *solutions* in the form $y = e^{kx}$, $k = cste$; so

$$y' = ke^{kx} \; ; \; y'' = k^2 e^{kx}.$$

Substitute these expressions into the equation E_h, we'll get:

$$e^{kx}(ak^2 + bk + c) = 0;$$

as $e^{kx} \neq 0$, we must have

$$ak^2 + bk + c = 0 \; -(eq).$$

NB: if y_1 and y_2 are solutions of the equation E_h then, by linearity, $y = c_1\,y_1 + c_2\,y_2$ is also a solution of the equation E_h. Indeed

$$\begin{aligned} ay'' + by' + cy &= a\,(c_1\,y_1 + c_2\,y_2)'' + b\,(c_1\,y_1 + c_2\,y_2)' + c\,(c_1\,y_1 + c_2\,y_2) \\ &= c_1\left[a\,y_1'' + b\,y_1' + c\,y_1\right] + c_2\left[a\,y_2'' + b\,y_2' + c\,y_2\right] \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Resolution (to remember):

*To solve the **linear homogeneous 2nd** order **ODE***

$$(a \neq 0) \quad ay'' + by' + cy = 0 \quad --E_h;$$

*we must consider its **characteristic equation***

$$ak^2 + bk + c = 0 \ -(eq)\ ;$$

and calculate the discriminant $\Delta = b^2 - 4ac$.

*1) If $\Delta > 0$: the characteristic equation (eq) admits **two distinct real roots*** $k_1 = \dfrac{-b-\sqrt{\Delta}}{2a}$ *and* $k_1 = \dfrac{-b+\sqrt{\Delta}}{2a}$;

the functions $y_1 = e^{k_1 x}$ *and* $y_2 = e^{k_2 x}$ *are **solutions of the equation*** E_h, *then:*

$$\boxed{\forall c_1, c_2 \in \mathbb{R} \quad : \quad y_H = c_1\, y_1 + c_2\, y_2 = c_1\, e^{k_1 x} + c_2\, e^{k_2 x}}$$

*is the **general solution** of the equation* E_h.

*2) If $\Delta = 0$: the characteristic equation (eq) admits a (**double**) **root***

$k = \dfrac{-b}{2a}$ *and the functions* $y_1 = e^{kx}$ *and* $y_2 = x\, e^{kx}$ *are **solutions of the equation*** E_h, *then :*

$$\boxed{\forall c_1, c_2 \in \mathbb{R} \quad : \quad y_H = c_1\, y_1 + c_2\, y_2 = \left(c_1 + c_2\, x\right) e^{kx}}$$

*is the **general solution** of the equation* E_h.

*3) If $\Delta < 0$: the characteristic equation (eq) admits **two conjugate complex roots***

$$k_1 = \frac{-b - i\sqrt{|\Delta|}}{2a} = \frac{-b}{2a} - i\,\frac{\sqrt{|\Delta|}}{2a} = \alpha - i\beta \quad \text{and} \quad k_1 = \frac{-b + i\sqrt{|\Delta|}}{2a} = \frac{-b}{2a} + i\,\frac{\sqrt{|\Delta|}}{2a} = \alpha + i\beta$$

*we deduce the existence of **complex solutions** of the equation* E_h.

$$y_1 = e^{(\alpha - i)\,x} = e^{\alpha\, x}\ e^{-i\,\beta\, x} \qquad \text{and} \qquad y_2 = e^{(\alpha + i)\,x} = e^{\alpha\, x}\ e^{+i\,\beta\, x}.$$

We can obtain ***real solutions***

$$\frac{y_2+y_1}{2} = e^{\alpha\, x}\ \frac{e^{+i\,\beta\, x} +\ e^{-i\,\beta\, x}}{2} = e^{\alpha\, x}\ \cos(\beta\, x)$$

and
$$\frac{y_2-y_1}{2i} = e^{\alpha\, x}\ \frac{e^{+i\,\beta\, x} -\ e^{-i\,\beta\, x}}{2i} = e^{\alpha\, x}\ \sin(\beta\, x)$$

then $\forall c_1, c_2 \in \mathbb{R}$:

$$\boxed{\begin{aligned} y_H &= c_1\, e^{\alpha\, x}\ \cos(\beta\, x) + c_2\, e^{\alpha\, x}\ \sin(\beta\, x) \\ &= e^{\alpha\, x}\ \left[c_1\ \cos(\beta\, x) + c_2\ \sin(\beta\, x)\right] \end{aligned}}$$

is the ***general solution*** *of the equation* E_h.

NB: Often in physics we rewrite this solution in the form

$\boxed{y_H = e^{\alpha\, x}\ A\ \cos\left(\beta\, x + \phi\right)}$ with $A = \sqrt{c_1^2 + c_2^2}$, $\cos\phi = \frac{c_1}{A}$, $\sin\phi = \frac{c_2}{A}$.

Example 1: let's solve the equation $y'' + y' - 2y = 0$.

characteristic equation : $k^2 + k - 2 = 0$; roots: $k_1 = 1$ and $k_2 = -2$,

general solution : $y = c_1\, e^{x} + c_2\, e^{-2x}$.

Example 2:

let's integrate the equation $y'' + 9y = 0$ with initial conditions $y_{x=0} = 0$, $y'_{x=0} = 3$.

characteristic equation: $k^2 + 9 = 0$; roots: $\Delta = -36$ $k_1 = 3i$ and $k_2 = -3i$.

general solution : $y = c_1\ \cos 3x + c_2\ \sin 3x$;

particular solution : we have $y' = -3c_1\ \sin 3x + 3c_2\ \cos 3x$. Applying the initial conditions, we'll get

$$\begin{matrix} y(0) = 0, \\ y'(0) = 3 \end{matrix} \iff \begin{matrix} 0 = C_1 \cos 0 + C_2 \sin 0, \\ 3 = -3C_1 \sin 0 + 3C_2 \cos 0 \end{matrix}$$

Calculate this we will obtain $C_1 = 0, C_2 = 1$.

Consequently, the *particular solution* is $y = \sin 3x$

Example 3: Solve $y'' + 2y' + 5y = 0$with $y(0) = 0$ and $y'(0) = 1$.

Characteristic equation : $k^2 + 2k + 5 = 0$, roots: $k_1 = -1 + 2i$ and $k_2 = -1 - 2i$;

complex solutions : $y = e^{-x}\, e^{\pm i\, 2x}$;

real solutions : $y_1 = e^{-x} \cos 2x$ and $y_2 = e^{-x} \sin 2x$;

general solution : $y = e^{-x}\big(c_1 \cos 2x + c_2 \sin 2x\big)$.

particular solution : $y' = e^{-x}\, 2c_2 \cos 2x + e^{-x}\, c_2 \sin 2x$

$$\begin{matrix} y(0) = 0, \\ y'(0) = 1 \end{matrix} \iff \begin{matrix} 0 = e^{0}\big(c_1 \cos 0 + c_2 \sin 0\big), \\ 1 = e^{-0}\, 2c_2 \cos 0 + e^{-0}\, c_2 \sin 0 \end{matrix}$$

We will get $c_1 = 0$ $c_2 = \frac{1}{2}$ then

The *particular solution* sought is $y = \dfrac{1}{2}\, e^{-x} \sin 2x$

Example 4: Integrate the equation $y'' - 4y' + 4y = 0$;

Characteristic equation: $k^2 - 4k + 4 = 0$; roots: $k_1 = k_2 = 2$.

general integral: $y = c_1\, e^{2x} + c_2\, x\, e^{2x}$.

Exercise

Solve the following differential equations. *Observe the curves of the solutions and compare. Explain the differences.*

1) $y'' + 9y = 0$ ***2)*** $y'' + y' + y = 0$

3) $16y'' - 8y' + 145y = 0$; $y(0) = -2$ and $y'(0) = 1$

Correction

1) Let the equation $y'' + 9y = 0$.

Characteristic equation: $k^2 + 9 = 0$;

roots: $k_1 = +3\, i$ and $k_2 = -3\, i$,

complex solutions: $y = e^{\pm i\, 3x}$;

real solutions: $y_1 = \cos 3x$ and $y_2 = \sin 3x$;

general integral : $\quad y = c_1 \cos 3x + c_2 \sin 3x$.

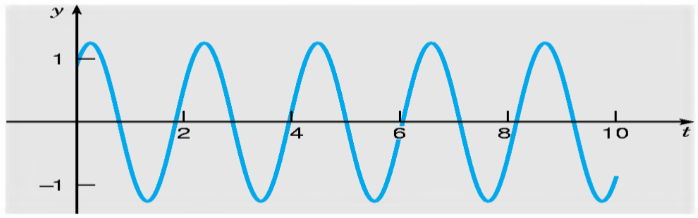


FIGURE 3.4.3 A typical solution of $y'' + 9y = 0$.

2) Let the equation $\quad y'' + y' + y = 0.$

Characteristic equation: $\quad k^2 + k + 1 = 0;$

roots: $\quad k_1 = \frac{1}{2} + i\,\frac{\sqrt{3}}{2}$and $k_2 = \frac{1}{2} - i\,\frac{\sqrt{3}}{2}$,

complex solutions $\quad y = e^{\frac{1}{2}x}\, e^{\pm i\,\frac{\sqrt{3}}{2}x};$

real solutions$y_1 = e^{-\frac{1}{2}x} \cos\dfrac{\sqrt{3}}{2}x$ and $y_2 = e^{-\frac{1}{2}x} \sin\dfrac{\sqrt{3}}{2}x;$

general integral : $\quad y = e^{-\frac{1}{2}x}\left(c_1 \cos\dfrac{\sqrt{3}}{2}x + c_2 \cos\dfrac{\sqrt{3}}{2}x\right).$

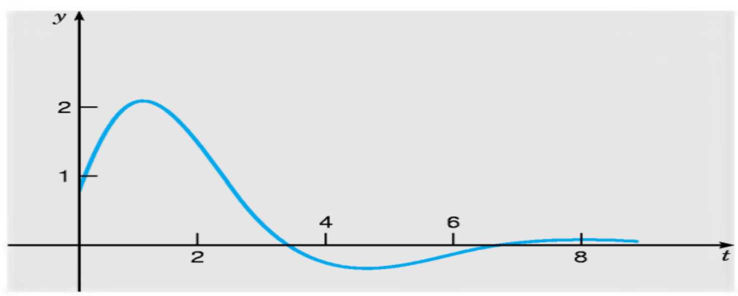


FIGURE 3.4.1 A typical solution of $y'' + y' + y = 0$.

3) Consider the equation: $16y'' - 8y' + 145y = 0; \quad y(0) = -2$ and $y'(0) = 1.$

Characteristic equation: $\quad 16k^2 - 8k + 145 = 0;$

roots: $\quad k_1 = \frac{1}{4} + 3\,i$and $k_2 = \frac{1}{4} + 3\,i$,

complex solutions: $y = e^{\frac{1}{4}x}\, e^{\pm i\,3x};$

real solutions : $\quad y_1 = e^{\frac{1}{4}x} \cos 3x$ and $y_2 = e^{\frac{1}{4}x} \sin 3x;$

general integral $\quad y = e^{\frac{1}{4}x}\left(c_1 \cos 3x + c_2 \sin 3x\right)$

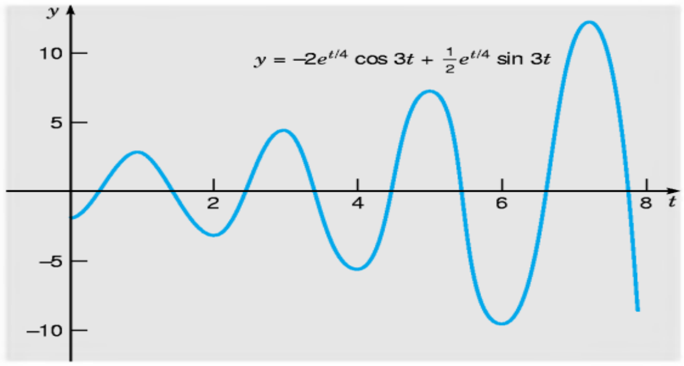
initial conditions : $\quad y' = e^{\frac{1}{4}x}\left[\left(\frac{1}{4}c_1 + 3c_2\right) \cos 3x + \left(\frac{1}{4}c_2 - 3c_1\right) \sin 3x\right];$

$$-2 = c_1 \cos 0 + c_2 \sin 0 = c_1$$

$$1 = \left(\frac{1}{4}c_1 + 3c_2\right) \cos 0 + \left(\frac{1}{4}c_2 - 3c_1\right) \sin 0 = \frac{1}{4}c_1 + 3c_2$$

we'll find $c_1 = -2$, $c_2 = \frac{1}{2}$.

The particular solution : $y = e^{\frac{1}{4}x}\left(-2\cos 3x + \frac{1}{2}\sin 3x\right)$



URE 3.4.2 Solution of $16y'' - 8y' + 145y = 0$, $y(0) = -2$, $y'(0) = 1$.

3.2. Inhomogeneous Linear Equation (with second member)

Theorem:

The general solution of the inhomogeneous equation (with second member):

$$ay'' + by' + cy = f(x)$$

is the sum of a ***particular solution*** y^* ***plus*** *the* ***general solution*** *y of the corresponding homogeneous equation*

$$ay'' + by' + cy = 0 \quad --E_h.$$

Question : *how to find a particular solution* y^**?*

3.2.1. Determination of coefficients

The second member allow to conjecture the form of the particular solution y^*(*see table*) and then we deal with ***the indeterminate coefficients*** method.

Whether $f(x) =$	Choose $y^* =$
$P_n(t)\ e^{\alpha t}$	$(A_0 + A_1x + A_2x^2 + \dots + A_nx^n)\ t^s\ e^{\alpha t}$
$P_n(t)\ e^{\alpha t}\ \cos\beta t$ or $P_n(t)\ e^{\alpha t}\ \sin\beta t$	$(A_0 + A_1x + A_2x^2 + \dots + A_nx^n)\ t^s\ e^{\alpha t}\ \cos\beta t$ $+$ $(A_0 + A_1x + A_2x^2 + \dots + A_nx^n)\ t^s\ e^{\alpha t}\ \sin\beta t$
where $s \in \{0,1\}$the smallest value for which $y^* = t^s \times (\dots)$is no longer a solution of the associated homogeneous equation.	

Example 1: Find the solutions of $y'' + 3y' - 4y = x^2$.

Let's find the solutions of the ***associated homogeneous*** equation :

$$y'' + 3y' - 4y = 0$$

characteristic equation $r^2 + 3r - 4 = 0$, $\Delta = 25$, $r_1 = 1$, $r_2 = -4$;

general solution : $y = c_1 e^t + c_2 e^{-4t}$.

Particular solution (*of the form of the second member*): $y^* = Ax^2 + Bx + C$.

Substituting y, $y^{*\,\prime} = 2Ax + B$ and $y^{*\,\prime\prime} = 2A$, into the equation we get

$$2A + 3(2Ax + B) - 4(Ax^2 + Bx + C) = x^2,$$

$$-4Ax^2 + (6A - 4B)x + 2A + 3B - 4C = x^2,$$

by ID: $-4A = 1$, $(6A - 4B) = 0$, $2A + 3B - 4C = 0$,

we deduce: $A = -\frac{1}{4}$, $B = -\frac{3}{8}$, $C = -\frac{13}{32}$.

The ***particular solution*** is $y^* = -\frac{1}{4}x^2 - \frac{3}{8}x - \frac{13}{32}x^2$

The ***solution of the inhomogeneous equation*** *is*

$$y = c_1 e^t + c_2 e^{-4t} - \frac{1}{4}x^2 - \frac{3}{8}x - \frac{13}{32}x^2.$$

Example 2: Find the solutions of $y'' - 3y' - 4y = 3e^{2t}$.

Let's find the solutions of the associated homogeneous equation :

$$y'' - 3y' - 4y = 0,$$

characteristic equation: $r^2 - 3r - 4 = 0$, $\Delta = 25$, $r_1 = -1$, $r_2 = 4$;

general solution : $y = c_1 e^{-t} + c_2 e^{4t}$.

Particular solution (of the form of the second member): $y^* = Ae^{2t}$;

we replace y, $y^{*\,\prime} = 2Ae^{2t}$ and $y^{*\,\prime\prime} = 4Ae^{2t}$, in the equation we'll get

$$4Ae^{2t} - 3(2Ae^{2t}) - 4(Ae^{2t}) = 3e^{2t},$$

$$-6Ae^{2t} = 3e^{2t} \qquad \text{and so} \qquad A = -\tfrac{1}{2}.$$

The ***particular solution*** is $y^* = -\frac{1}{2}e^{2t}$.

The ***solution of the inhomogeneous equation*** is

$$y = c_1 e^{-t} + c_2 e^{4t} - \frac{1}{2}e^{2t}.$$

Example 3: Find the solutions of $y'' - 3y' - 4y = 2\sin t$.

Let's find the solutions of the associated homogeneous equation :

$$y'' - 3y' - 4y = 0,$$

characteristic equation: $r^2 - 3r - 4 = 0$, $\Delta = 25$, $r_1 = -1$, $r_2 = 4$,

general solution: $y = c_1 e^{-t} + c_2 e^{4t}$.

Particular solution (*of the form of the second member*): $y^* = A\sin t + B\cos t$.

By substituting y, $y^{*\,\prime} = A\cos t - B\sin t$ and ,
$y^{*\,\prime\prime} = -A\sin t - B\cos t$, we'll get

$$(-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) = 2\sin t$$

$$(3B - 5A)\sin t + (-3A - 5B)\cos t = 2\sin t,$$

then $3B - 5A = 2$ and $-3A - 5B = 0$, we deduce $A = -5/17$ and $B = 3/17$.

The ***particular solution*** is $y^* = -5/17\sin t + 3/17\cos t$

The ***solution of the inhomogeneous equation*** is

$$y = c_1 e^{-t} + c_2 e^{4t} - 5/17\sin t + 3/17\cos t.$$

Example 4: Find the solutions of $y'' + 4y = 3\sin 2t$.

Let's find the solutions of the associated homogeneous equation :

$$y'' + 4y = 0$$

characteristic equation: $r^2 + 4 = 0$, $r_1 = -2i$, $r_2 = 2i$.

general solution: $y = c_1\cos 2t + c_2\sin 2t$.

Particular solution (*of the form of the second member*): $y^* = A\sin 2t + B\cos 2t$.

substituting y, $y^{*\,\prime} = 2A\cos 2t - 2B\sin 2t$, $y^{*\,\prime\prime} = -4A\sin 2t - 4B\cos 2t$, we'll get

$$(-4A\sin 2t - 4B\cos 2t) + 4(A\sin 2t + B\cos 2t) = 3\sin 2t,$$

$$(-4A + 4A)\sin 2t + (-4B + 4B)\cos 2t = 3\sin 2t$$

Remarque : By identification we notice that **we cannot find such A and B.** $y^* = A\sin 2t + B\cos 2t$ is solution of homogeneous equation.

> ***NB (important):***
> *If the **chosen form** y^* **is already a solution of the associated homogeneous equation**, we must seek a **particular solution in the form***
> $$y^* = t^s\ (A\sin 2t + B\cos 2t)$$
> *with $s \in \{1.2\}$ **such that** $y^*$is **no longer a solution** of the associated homogeneous equation.*

Therefore, we must look for a particular solution of the form

$$y^* = A\ t\ \sin 2t + B\ t\ \cos 2t.$$

We calculate $y^{*\,\prime} = \ldots$and $y^{*\,\prime\prime} = \ldots$, we substitute in the equation to obtain,

$-4A\sin 2t + 4B\cos 2t = 3\sin 2t$ then $B = 0$ and $A = -3/4$.

The ***particular solution*** is $y^* = -4/3\ t\ \sin 2t$

The ***solution of the inhomogeneous equation*** is

$$y = c_1\cos 2t + c_2\sin 2t - 4/3\ t\ \sin 2t.$$

3.2.2. Variation of the constant

This method has the advantage of being general, it applies to any kind of differential equation, but the calculations are more consistent.

> ***Method:***
> *We take the general solution of the homogeneous equation (Eh) and we transform the constants C_1 and C_2 into functions $C_1(.)$ and $C_2(t)$...*

Example : **1)** Check that $\displaystyle\int \frac{1}{\sin t}dx = \ln\left[\frac{1}{\sin t} - \frac{\cos x}{\sin t}\right]$

2) Find the solutions of $y'' + 4y = \frac{3}{\sin t}$.

1) let us set $u := \dfrac{1}{\sin t} - \dfrac{\cos x}{\sin t}$ and note that

$$\int \frac{1}{\sin t}dx = \ln\left[\frac{1}{\sin t} - \frac{\cos x}{\sin t}\right] = \ln u \iff \frac{u'}{u} = \frac{1}{\sin t}.$$

$$u' = \left[\frac{1}{\sin t} - \frac{\cos x}{\sin t}\right]' = \frac{-\cos x}{\sin^2 t} - \frac{-1}{\sin^2 t} = \frac{1-\cos x}{\sin^2 t},$$

$$\frac{u'}{u} = \frac{1-\cos x}{\sin^2 t} : \frac{1-\cos x}{\sin t} = \frac{1-\cos x}{\sin^2 t} . \frac{\sin t}{1-\cos x} = \frac{1}{\sin t}.$$

2) Let's find the solutions of the associated homogeneous equation:

$$y'' + 4y = 0,$$

characteristic equation: $r^2 + 4 = 0$, $r_1 = -2i$, $r_2 = 2i$,

general solution: $y = c_1 \cos 2t + c_2 \sin 2t$.

Let's use the constant variation method : Determine $c_1(t)$ *et* $c_2(t)$ such that

$$y = c_1(t)\ \cos 2t + c_2(t)\ \sin 2t$$

is solution of the inhomogeneous equation $y'' + 4y = \frac{3}{\sin t}$.

By differentiating we get

$$y' = -2c_1(t)\ \sin 2t + 2c_2(t)\ \cos 2t + c_1'(t)\ \cos 2t + c_2'(t)\ \sin 2t.$$

Let us choose for simplicity $c_1(t)$ *et* $c_2(t)$ such that

$$c_1'(t)\ \cos 2t + c_2'(t)\ \sin 2t = 0 \quad \textbf{(A)}$$

then we will have

$$y' = -2c_1(t)\ \sin 2t + 2c_2(t)\ \cos 2t \qquad \text{(I)}$$

A second derivation gives

$$y'' = -4c_1(t)\ \cos 2t - 4c_2(t)\ \sin 2t - 2c_1'(t)\ \sin 2t + 2c_2'(t)\ \cos 2t \qquad \text{(II)}$$

By replacing (I) and (II) in the equation we obtain after calculation

$$-2c_1'(t)\ \sin 2t + 2c_2'(t)\ \cos 2t = \frac{3}{\sin t} \quad \textbf{(B)}$$

In summary $c_1(t)$ *et* $c_2(t)$ must verify the equations (A) and (B):

$$c_1'(t)\ \cos 2t + c_2'(t)\ \sin 2t = 0 \qquad (A)$$

$$2c_1'(t)\ \sin 2t + 2c_2'(t)\ \cos 2t = \frac{3}{\sin t} \qquad (B)$$

From (A) we have: $c_2'(t) = -c_1'(t)\frac{\cos 2t}{\sin 2t}$ (C)

We replace in (B): $c_1'(t) = -\dfrac{3\sin 2t}{2\sin t} = -3\cos t$

by integrating $c_1(t) = -3\sin t + k_1$

Resuming $c_1'(t)$ in (C):

$$c_2'(t) = \frac{3\cos t\ \cos 2t}{\sin 2t} = \frac{3(1-2\sin^2 t)}{2\sin t} = \frac{3}{2\sin t} - 3\sin t$$

by integrating $c_2(t) = \dfrac{3}{2}\ln\left[\dfrac{1}{\sin t} - \dfrac{\cos t}{\sin t}\right] + 3\cos t + k_2$.

Consequently the ***solution of the inhomogeneous equation is***

$$y = (-3\sin t + k_1)\cos 2t + (\frac{3}{2}\ln\left[\frac{1}{\sin t} - \frac{\cos t}{\sin t}\right] + 3\cos t + k_2)\sin 2t$$

$$y = -3\sin t\cos 2t + 3\cos t\sin 2t + \frac{3}{2}\ln\left[\frac{1}{\sin t} - \frac{\cos t}{\sin t}\right]\sin 2t + k_1\cos 2t + k_2\sin 2t$$

$$y = -3\sin t[2\cos^2 t - 1] + 6\cos^2 t\sin t + \frac{3}{2}\ln\left[\frac{1}{\sin t} - \frac{\cos t}{\sin t}\right]\sin 2t + k_1\cos 2t + k_2\sin 2t$$

The solution is finally

$$y = 3\sin t + \frac{3}{2}\ln\left[\frac{1}{\sin t} - \frac{\cos t}{\sin t}\right]\sin 2t + k_1\cos 2t + k_2\sin 2t$$

4 HIGH ORDER EQUATION

1) Solve $y^{(4)} + y''' - 7y'' - y' + 6y = 0$,

with the initial conditions $y(0) = 1\ y'(0) = 0\ y''(0) = -2\ y'''(0) = -1$.

Solution :

characteristic equation: $k^4 + k^3 - 7k^2 - k + 6 = 0$;

roots: let's check for $k = \pm 1$, we have

$1^4 + 1^3 - 7x1^2 - 1 + 6 = 1 + 1 - 7 - 1 + 6 = 0$ then $k_1 = 1$ is a root.

$(-1)^4 + (-1)^3 - 7x(-1)^2 - (-1) + 6 = 1 - 1 - 7 + 1 + 6 = 0$ then $k_2 = -1$ is a root.

We deduce

$$k^4 + k^3 - 7k^2 - k + 6 = (k-1)(k+1)P(k),$$

and after calculation:

$$k^4 + k^3 - 7k^2 - k + 6 = (k-1)(k+1)(k^2 + k - 6);$$

the ***roots are:*** $k_1 = 1 \qquad k_2 = -1 \qquad k_3 = 2 \qquad k_4 = -3.$

Solutions: $y_1 = e^x \qquad y_2 = e^{-x} \qquad y_3 = e^{2x} \qquad y_4 = e^{-3x}.$

General integral: $y = C_1\, e^x + C_2\, e^{-x} + C_3\, e^{2x} + C_4\, e^{-3x}.$

Particular solution: we need the derivatives

$$y' = C_1\, e^x - C_2\, e^{-x} + 2C_3\, e^{2x} - 3C_4\, e^{-3x}$$

$$y'' = C_1\, e^x + C_2\, e^{-x} + 4C_3\, e^{2x} + 9C_4\, e^{-3x}$$

$$y''' = C_1\, e^x - C_2\, e^{-x} + 8C_3\, e^{2x} - 27C_4\, e^{-3x},$$

thus

$$\begin{cases} y(0) = 1 \\ y'(0) = 0 \\ y''(0) = -2 \\ y'''(0) = -1 \end{cases} \iff \begin{cases} C_1 + C_2 + C_3 + C_4 = 0 \\ C_1 - C_2 + 2C_3 - 3C_4 = 0 \\ C_1 + C_2 + 4C_3 + 9C_4 = 0 \\ C_1 - C_2 + 8C_3 - 27C_4 = 0 \end{cases}.$$

Solving the system we get:

$$C_1 = \frac{11}{8} \qquad C_2 = \frac{5}{12} \qquad C_3 = \frac{-2}{3} \qquad C_4 = \frac{-1}{8}.$$

Hence the ***particular solution is :*** $y = \dfrac{11}{8}\, e^x + \dfrac{5}{12}\, e^{-x} - \dfrac{2}{3}\, e^{2x} - \dfrac{1}{8}\, e^{-3x}.$

2) Solve $y^{(4)} - y = 0$
with the initial conditions $y(0) = \frac{7}{2}\; y'(0) = -4\; y''(0) = \frac{5}{2}\; y'''(0) = -2.$

<u>Solution :</u>

characteristic equation: $r^4 - 1 = (r^2 - 1)(r^2 + 1) = 0$;

the ***roots:*** $k_1 = 1 \qquad k_2 = -1 \qquad k_3 = i \qquad k_4 = -i.$

solution : $y_1 = e^x \qquad y_2 = e^{-x} \qquad y_3 = \cos x \qquad y_4 = \sin x.$

General integral: $y = C_1\, e^x + C_2\, e^{-x} + C_3\, \cos x + C_4\, \sin x.$

particular solution: we need the derivatives

$$y' = C_1\, e^x - C_2\, e^{-x} - C_3\, \sin x + C_4\, \cos x$$

$$y'' = C_1\, e^x + C_2\, e^{-x} - C_3\, \cos x - C_4\, \sin x$$

$$y''' = C_1\, e^x - C_2\, e^{-x} + C_3\, \sin x + C_4\, \cos x$$

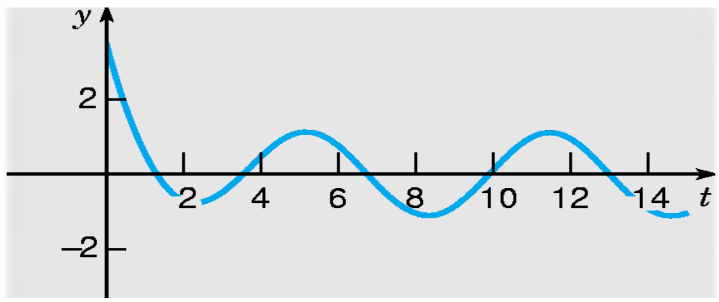
thus

$$\begin{cases} y(0)=\frac{7}{2} \\ y'(0)=-4 \\ y''(0)=\frac{5}{2} \\ y'''(0)=-2y(0)=1 \end{cases} \iff \begin{cases} C_1+C_2+C_3=\frac{7}{2} \\ C_1-C_2+C_4=-4 \\ C_1+C_2-C_3=\frac{5}{2} \\ C_1-C_2-C_4=1 \end{cases}$$

Solving the system we get:

$$C_1=0 \qquad C_2=3 \quad C_3=\frac{1}{2} \qquad C_4=-1..$$

Hence the ***particular solution is :*** $\quad y=3\,e^{-x}+\dfrac{1}{2}\cos x-\sin x.$



3) Solve$y^{(4)}+2y''+y=3\sin x-5\cos x$

<u>Solution :</u>

characteristic equation: $\quad r^4+r^2+1=(r^2+1)(r^2+1)=0$;

the ***roots***: $\quad r_1=i \qquad r_2=-i \qquad r_3=i \qquad r_4=-i.$

solutions of the homogeneous equation :

$$y_1=\cos x \qquad y_2=\sin x \qquad y_3=x\ \cos x \quad y_4=x\ \sin x.$$

Solution general of the homogeneous equation :

$$y_h=C_1\ \cos x+C_2\ \sin x+C_3\ x\ \cos x+C_4\ x\ \sin x.$$

Particular solution: it must be in the form $y^*=A\sin x+B\cos x$

but since ***we already have the solutions*** $y_3=x\ \cos x$ and $y_4=x\ \sin x$

we must ***look for the solution*** in the form $y^*=Ax^2\ \sin x+Bx^2\ \cos x$.

By differentiating and then substituting in the equation we get

$$-8x^2\sin x-B\cos x=3\sin x-5\cos x$$

and by identification we obtain $A = \frac{-3}{8}$ and $B = \frac{5}{8}$, hence

the ***general solution is***:

$$y = C_1 \cos x + C_2 \sin x + C_3 \; x \cos x + C_4 \; x \sin x \frac{3}{8} x^2 \sin x + \frac{5}{8} x^2 \cos x$$

4) Solve $y''' - 4y' = t + 3\cos t + e^{-2t}$

Solution :

characteristic equation: $r^3 - 4r = r(r^2 - 4) = 0$;

roots: $r_1 = 0$ $r_2 = 2$ $r_3 = -2$.

Solutions of the homogeneous equation :

$$y_1 = e^{0\,t} = 1 \qquad y_2 = e^{2\,t} \qquad y_3 = e^{-2\,t}.$$

General solution of the homogeneous equation :

$$y_h = C - 1 + C_2 \, e^{2t} + C_3 \, e^{-2t}.$$

Special solution:

The second member of the equation is a sum so we can simplify the calculations by considering

$$y^* = y_1^* + y_2^* + y_3^*$$

where

a) y_1^* will be ***particular solution*** of $y''' - 4y' = t$.

Let us pose $y_1^* = c_1 t + c_2$, but as $y_1 = cste$ is already solution then one must seeks $y_1^* = t(c_1 t + c_2)$.

b) y_2^* will be ***particular solution*** of $y''' - 4y' = 3\cos t$. Let's put $y_2^* = c_3 \cos t + c_4 \sin t$.

c) y_3^* will be ***particular solution*** of $y''' - 4y' = e^{-2t}$.

Let's put $y_3^* = c_5 e^{-2t}$ but as $y_3 = e^{-2\,t}$ is already a solution then we have to seek $y_3^* = c_5 \, te^{-2t}$.

The resolution of each case will give

$$c_1 = \tfrac{-1}{8} \; , \; c_2 = 0 \; , \; c_3 = 0 \; , \; c_4 = \tfrac{-3}{5} \; , \; c_5 = \tfrac{1}{8} \; ,$$

and $y^* = y_1^* + y_2^* + y_3^* = -\frac{1}{8} t^2 - \frac{3}{5} \sin t + \frac{1}{8} \, te^{-2t}$.

the ***general solution is***:

$$y = C - 1 + C_2 \, e^{2t} + C_3 \, e^{-2t} + y^* = C - 1 + C_2 \, e^{2t} + C_3 \, e^{-2t} - \frac{1}{8} t^2 - \frac{3}{5} \sin t + \frac{1}{8} te^{-2t}.$$

Chap 4 : Functions with several variables

1 GENERALITIES

1.1. Introduction

The functions of several variables are natural, for examples of that:

1) temperature depends on latitude, longitude and time:

$$T : \mathbb{R}^3 \to \mathbb{R}$$
$$(x, y, t) \to T(x, y, t)$$

2) the cost of an advertising brochure depends on its format (A4, A5), the number of pages, the number of colors used, etc.

Definition

- ✓ *A function f of $\mathbb{R}^n$ with real values is a relation that corresponds to any point $X = (x_1, x_2, ..., x_n)$ of $\mathbb{R}^n$ at most one real number $f(X)$.*
- ✓ *The **domain of definition** of f is the set $D_f \subset \mathbb{R}^n$ of points $X = (x_1, x_2, ..., x_n)$ which have an image by f.*
- ✓ *The **image** by f of D is the set $Imf(D_f) = \{r = f(X) \ , \ X \in \mathbb{R}^n\} \subset \mathbb{R}$.*
- ✓ *The set of points $S = \left\{(X, f(X)) \ , \ X\right\} \subset \mathbb{R}^{n+1}$ is the **representative curve** of f.*

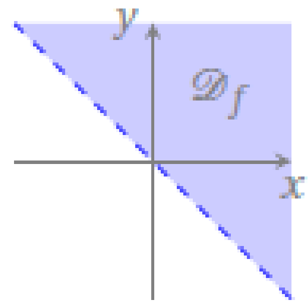
N.B.: we often use the notations (x, y) if *n*= 2 and (x, y, z) if *n*= 3.

Example

The domain of definition of the function $f(x,y) = \sqrt{x+y}$ is given by $D_f = \{(x,y) \in \mathbb{R}^2 : x+y \geq 0\}$. It is represented in a half-plane.

In addition, the values taken by the function go through the entire set of positive or zero real numbers:

$$Imf(D_f) = \mathbb{R}_+.$$

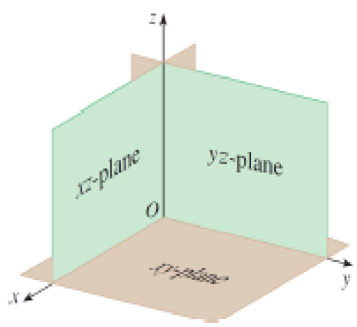


N.B.:

The geometric representation becomes heavier than for functions of a single variable (***n variables are visualized a priori in a space with n +1 dimensions***).

1.2. Functions of two variables

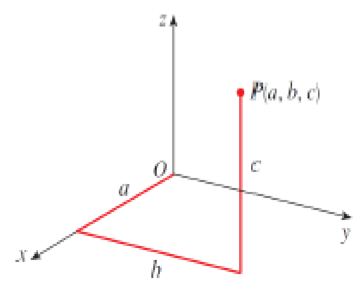
When **n = 2** , the **graph** $G_f = \{(x,y,z) : (x,y) \in \mathbb{R}^2 \ et \ z = f(x,y)\}$ is **three-dimensional** . The axes relating to the variables, **x and y** , are conventionally located in a **horizontal plane** (***the domain D then appears as a subset of this plane***), while the **vertical dimension** is reserved for the **values of z** .



the *xy* plane equation: z=0;

the *xz* plane equation: y =0;

the *yz* plane equation: x =0



Exercise

Determine and represent the domain of definition of the functions given by:

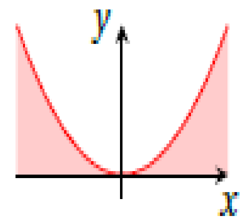
$$1) f(x,y) = \frac{\sqrt{-y+x^2}}{\sqrt{y}}\ ,\ 2) f(x,y) = \frac{\ln y}{\sqrt{x-y}}\ ,\ 3) f(x,y) = \ln(x+y)\ ,\ 4) f(x,y) = \frac{\ln(x^2+1)}{x\,y}.$$

Correction

1) $f(x,y)=\dfrac{\sqrt{-y+x^2}}{\sqrt{y}}$

$x\in D_f \iff \begin{cases} -y+x^2\geq 0 \\ y>0 \end{cases} \iff \begin{cases} y\leq x^2 \\ y>0 \end{cases}$

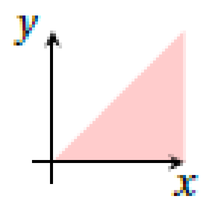
This is the intersection of the positive y half-plane the lower part of the parabola of equation $y=x^2$.



2) $f(x,y)=\dfrac{\ln y}{\sqrt{x-y}}$

$x\in D_f \iff \begin{cases} y>0 \\ x-y>0 \end{cases} \iff \begin{cases} y>0 \\ y<x \end{cases}$

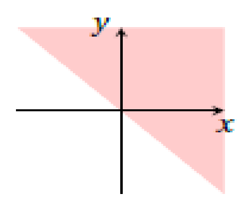
This is the intersection of the positive y half-plane the lower part of the line of equation $y=x$.



3) $f(x,y)=\ln(x+y)$

$x\in D_f \iff x+y>0 \iff y>-x$

This is the intersection of the positive y half-plane the lower part of the line of equation $y=-x$.

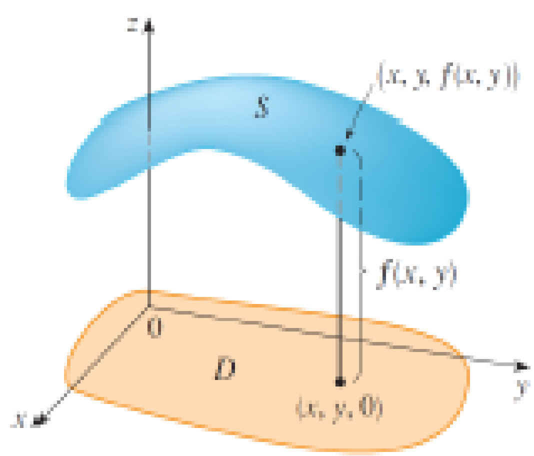


4) $f(x,y)=\dfrac{\ln(x^2+1)}{x\,y}$, $x\in D_f \iff x\neq 0 \text{ et } y\neq 0$. This is the entire plane deprived of the origin.

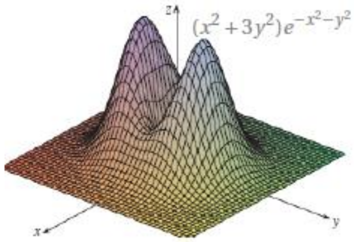
1.2.1. Surface representation

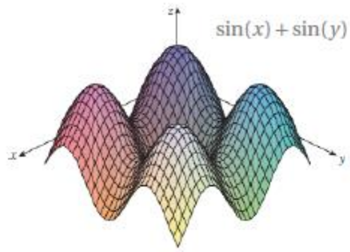
The altitude $z=f(x,y)$ is used to illustrate the graph of the function $f:\mathbb{R}^2\to\mathbb{R}$.

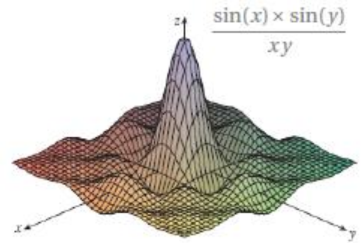
$(x,y)\to f(x,y)$.



Example :



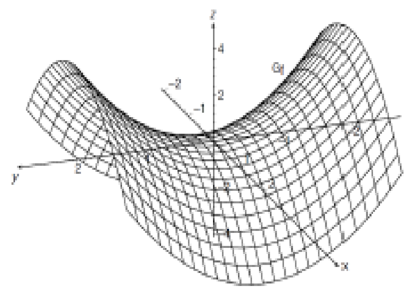




Example : (***Horse saddle***) The graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

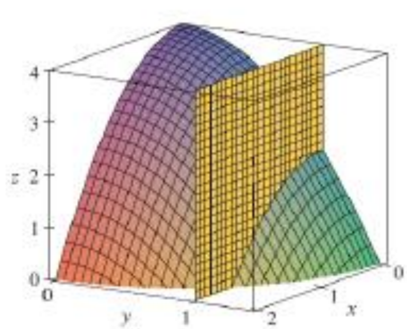
$$f(x, y) = x^2 - y^2$$

is a surface that has the form of a **horse's saddle**.



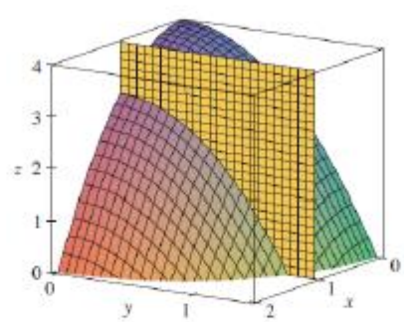
1.2.2. Partial functions

The partial functions associated with $f : \mathbb{R}^2 \to \mathbb{R}$ are functions of $\mathbb{R}$ in $\mathbb{R}$ given by **the intersection** of the **representative surface** of f *with* " vertical **planes parallel to the axes** ".



$f_b : \mathbb{R} \to \mathbb{R}$

$x \mapsto f(x, b)$



$f_a : \mathbb{R} \to \mathbb{R}$

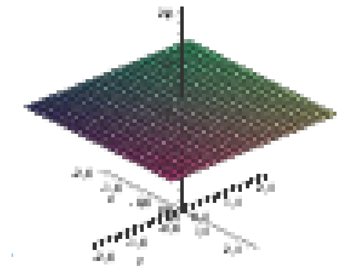
$y \mapsto f(a, y)$

Example : Thanks to partial functions we can guess the surface representation of $f : \mathbb{R}^2 \to \mathbb{R}$ simple functions:

1) If $f(x,y)=1$ then

$$\begin{cases} f(x,b)=1 & line \\ f(a,y)=1 & line \end{cases}$$

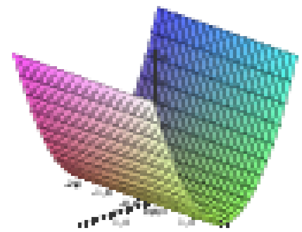
thus, the representative curve de f is as follow



2) If $f(x,y)=x^2$ then

$$\begin{cases} f(x,b)=x^2 & parbola \\ f(a,y)=cste & line \end{cases}$$

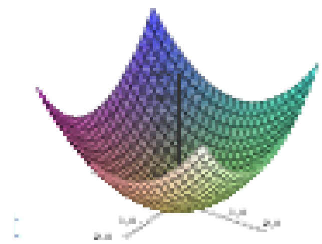
thus, the representative curve de f is as follow



3) If $f(x,y)=x^2+y^2$ then

$$\begin{cases} f(x,b)=x^2 & parbola \\ f(a,y)=y^2 & line \end{cases}$$

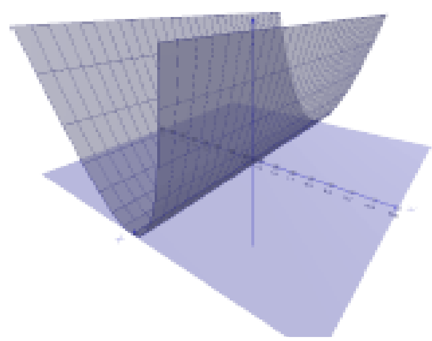
thus, the representative curve de f is as follow



Exercise

1) Guess the expression of a function whose **iso-0** (i.e. the level 0 line) represents the bank of a straight river.

2) Modify the previous function so that the water flows in the direction of positive x



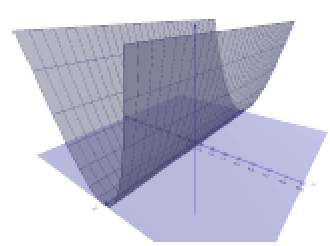
Correction

1) We have $f(a,y)=y^2 \quad (parbola)$

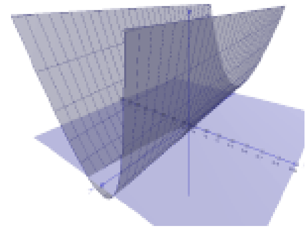
and $f(x,b)=cste \quad (line)$

therefore

$$f(x,y)=y^2.$$



2) For water to flow in the direction of positive x we must have $$f(x,b) = ax \quad (a < 0),$$



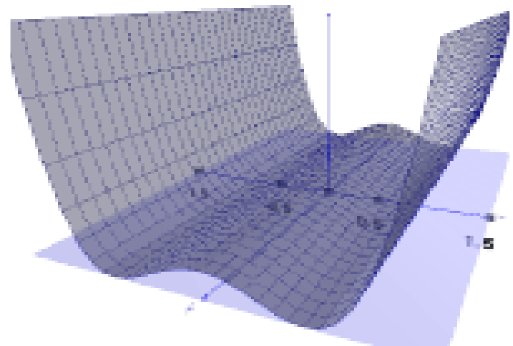
i.e. a line inclined towards the positive x, and

$f(a,y) = y^2 \quad (parbola).$

We deduce $$f(x,y) = y^2 + ax \ (a < 0).$$

$f(x,y) = y^2 - \frac{1}{10}x$ for example, for a slope of 10%.

Exercise

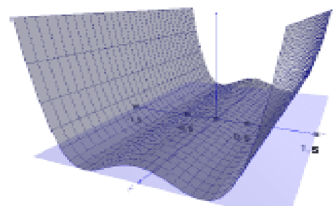


1) Guess the expression of a function whose **iso-0** (i.e. the level 0 line) represents the bank of a straight river.

2) Modify the previous function so that the water flows in the direction of positive x

Correction

1) We have $f(x,b) = cste \quad (line)$

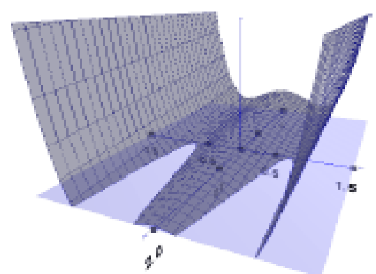


et a curve parallel to a y with a form  ,

thus $$f(a,y) = y^4 - y^2.$$

Therefore $$f(x,y) = y^4 - y^2.$$

2) For water to flow in the direction of positive x we must have $f(x,b) = ax \quad (a < 0),$



i.e. a line inclined towards the positive x, and
$$f(a,y) = y^4 - y^2.$$

We deduce $f(x,y) = y^4 - y^2 + ax \ (a < 0).$

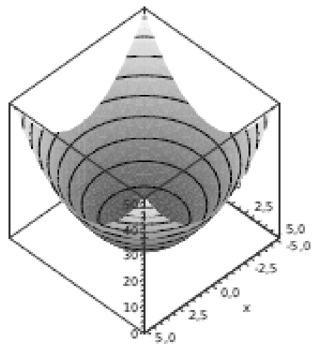
$f(x,y) = y^4 - y^2 - \frac{1}{10}x$ for example, for a slope of 10%.

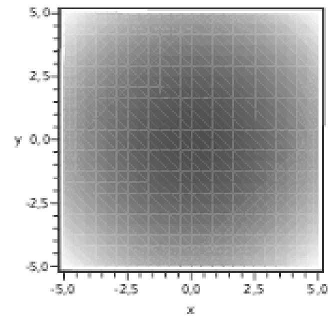
1.2.3. Planar representation

The shades of gray in a black and white photo are the representation of a function defined on a rectangle with values in the interval [0; 1]: 0 black, 1 white. We speak of **planar representation**.

Example

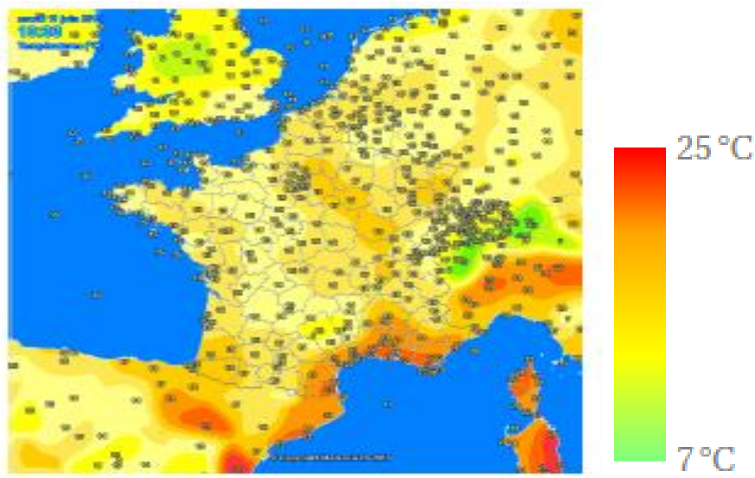
Surface and planar representation of the function $(x,y) \to x^2 + y^2$.





Color shades are also used in planar representation.

<u>Example</u> **(** ***Weather Maps*** **)**



1.2.4. Representation by level lines

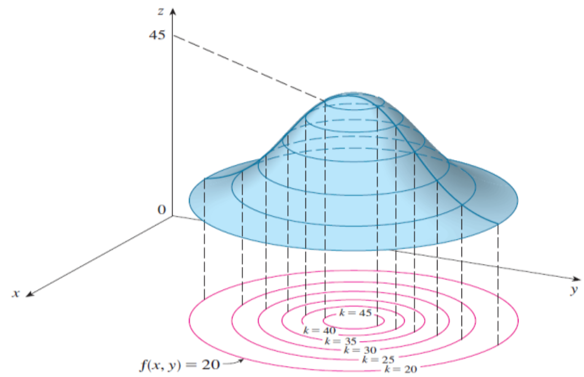
Recall that to obtain the **partial functions**, we considered **vertical cuts** of the graph of a function of two variables.

In the same way, we can consider **horizontal cuts** to obtain plane curves, called curves or **level lines**.

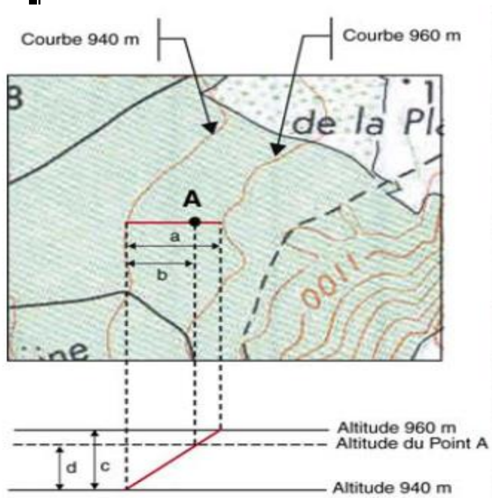
> ***<u>Definition (Level lines):</u>***
> *Let $K \in \mathbb{R}$ And a function $f : \mathbb{R}^2 \to \mathbb{R}$; the* **level curve** K *of f is the* ***projection onto the equation plane*** $z = \mathbf{0}$ *(plane of (x,y)) of the* ***intersection of the representative surface*** *of f with the* ***horizontal plane*** $z = k$, **i.e.** $\{(x, y, z) : (x, y) \in D_f \ : \ f(x, y) = K\}$.

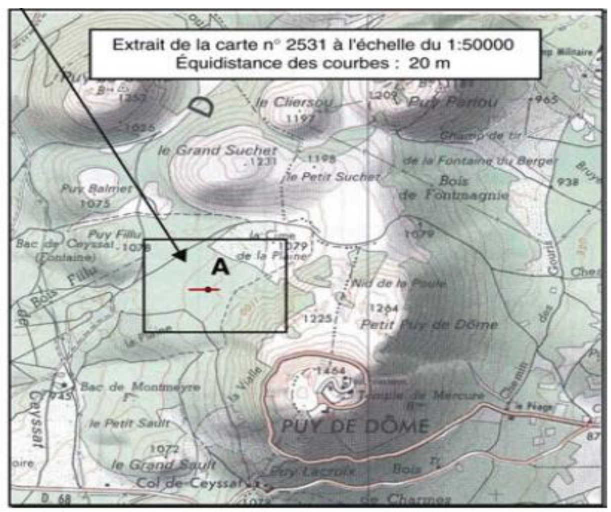
In practice, different level curves are represented simultaneously to visualize the progression of the graph.

This representation is similar to geographical maps where the level corresponds to the altitude.



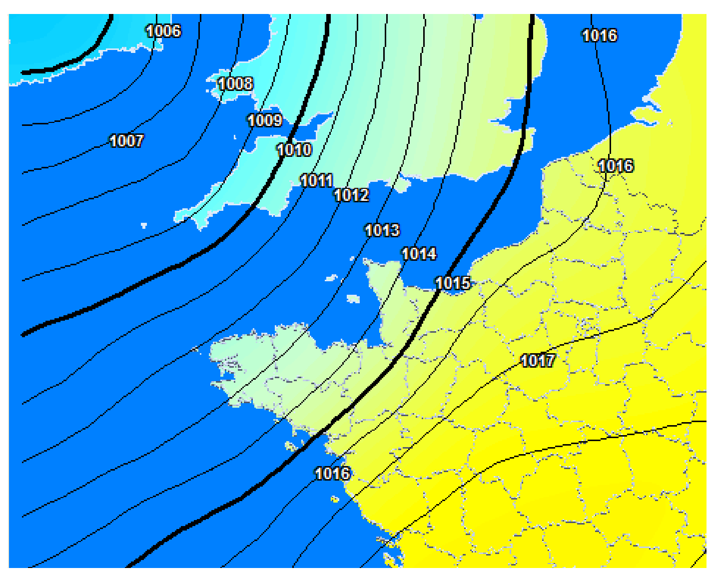
> ***Example* (Topographic maps)**
> *In the relief of a region, a contour curve indicates points of the same altitude. By drawing the contour lines with their corresponding* **altitude**, *we obtain the* ***topographic relief map***





Exemple : (***Weather maps***)

On a weather map, the contour lines are **isotherms** (**lines connecting points of equal temperature**); or **isobars** (**lines connecting points of equal pressure**).



Exercise

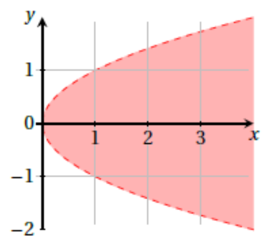
1) Determine and represent the domain of definition of the function $f:\mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y)=\ln(x-y^2)$.

2) Determine and represent its contour lines.

Correction

1) $f(x,y)=\ln(x-y^2)$

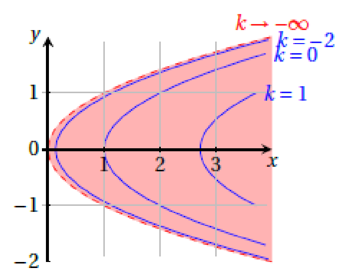
$$x \in D_f \iff x-y^2>0$$
$$\iff -\sqrt{x}<y<+\sqrt{x}.$$



2)

$$f(x,y)=k \iff x-y^2=e^k$$
$$\iff -\sqrt{y}=\pm\sqrt{x-C} \quad (C=e^k>0).$$

These are translated to the right of the curve defined by $y=\pm\sqrt{x}$.



2 LIMITS OF A FUNCTION

We recall the "**Euclidean** " **distance** defined in $\mathbb{R}^n$(n=2,3) by

n=2: $x=(x_1,x_2)$, $y=(y_1,y_2)$ $\in \mathbb{R}^2$ $\quad d(x,y):=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$

n =3: $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)$ $\in \mathbb{R}^2$ $\quad d(x,y):=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}$

> ***Limit at a point in $\mathbb{R}^n$:***
>
> *Let be $f : D_f \subset \mathbb{R}^n \to \mathbb{R}$ a function and $L \in \mathbb{R}$.*
>
> *We say that the limit of f when x tends to $X_0 \in \mathbb{R}^n$ is equal to L (we write $\lim_{X\to X_0} f(X)=L$ or $f(X) \xrightarrow[X\to X_0]{} L$) if:*
>
> *for everything $\varepsilon>0$, there exists $\delta>0$ such as $d(X,X_0)<\delta \implies d(f(X),L)<\varepsilon$.*
>
> ***This limit may exist even if f is not defined in x_0.***

We say that the limit of f when X tends to $X_0 \in \mathbb{R}^n$ is equal to $+\infty$ (we write $\lim_{X \to X_0} f(X) = +\infty$ or $f(X) \xrightarrow[X \to X_0]{} L$) if:

for everything $A > 0$, there exists $\delta > 0$ such as $d(X, X_0) < \delta \implies f(X) > A$.

We say that the limit of f when X tends to $X_0 \in \mathbb{R}^n$ is equal to $-\infty$ (we write $\lim_{X \to X_0} f(X) = -\infty$ or $f(X) \xrightarrow[X \to X_0]{} -\infty$) if:

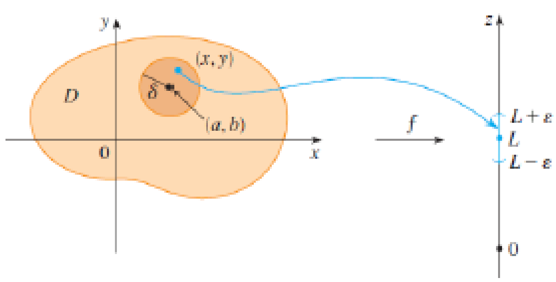
for everything $A > 0$, there exists $\delta > 0$ such as $d(X, X_0) < \delta \implies f(X) < -A$.

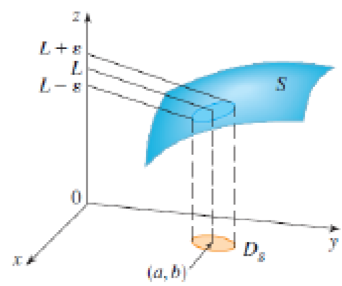
Exercise

By copying from the previous definition give those of:

1) $\begin{array}{l} f : \mathbb{R}^2 \to \mathbb{R} \\ (x, y) \to z = f(x, y) \end{array}$ 2) $\begin{array}{l} f : \mathbb{R}^3 \to \mathbb{R} \\ X = (x_1, x_2, x_3) \to f(X) \end{array}$

Correction





1) $\lim_{(x,y) \to (a,b)} f(x, y) = L$ or $f(x, y) \xrightarrow[(x,y) \to (a,b)]{} L$ *if:*

$\forall\, \varepsilon > 0$, it exists $\delta > 0$ such as

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x, y) - L| < \varepsilon.$$

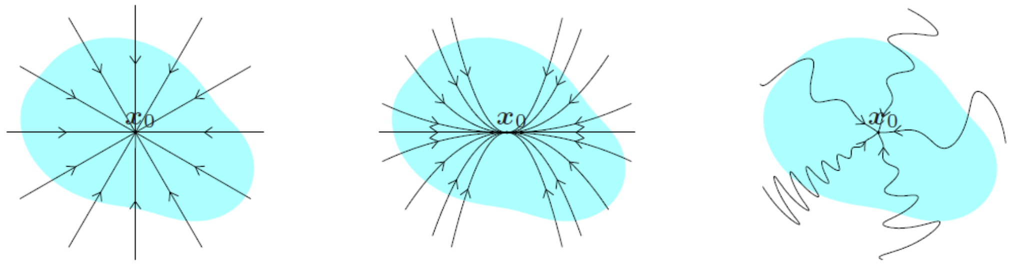
2) $\lim_{X \to (a,b,c)} f(X) = L$ or $f(X) \xrightarrow[X \to (a,b,c)]{} L$ *if:*

$\forall\, \varepsilon > 0$, it exists $\delta > 0$ such as

$$\sqrt{(x_1-a)^2 + (x_2-b)^2 + (x_3-b)^2} < \delta \implies |f(X) - L| < \varepsilon.$$

N.B.:

Unfortunately, it's not simple anymore for $\mathbb{R}^n$, $n \geq 2$, because there are an infinite number of possible directions to go towards a point $X_0 \in \mathbb{R}^n$.



> ***Proposition:***
>
> *Let $f\ :\ \mathbb{R}^n \to \mathbb{R}$ be a function.*
>
> $\lim\limits_{X\to X_0} f(X) = L$ *if and only if* $\lim\limits_{X \xrightarrow{X\in D} X_0} f(X) = L$ *for **any direction D***
>
> *(curve in $\mathbb{R}^n$passing through X_0).*

Example

We want to calculate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2} = ?$

Correction

1) *the domain of function* $f:(x,y)\to f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ *is the set* $\mathbb{R}^2 \setminus \{(0,0)\}$.

2) *We look for continuous curves defined on* $\mathbb{R}^2$*, which pass through* $(0,0)$*, and we calculate the limit of the restriction of* f *to these curves:*

i) *on the line of equation* $y=0$ *(x axis) we have:*

$$g(x) := f(x,0) = \frac{x^2-0^2}{x^2+0^2} = \frac{x^2}{x^2} = 1$$

we deduce $$\lim_{(x,y)\xrightarrow{y=0}(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} g(x) = 1$$

ii) *on the line of equation* $x=0$ *(y axis) we have:*

$$h(y) := f(0,y) = \frac{0^2 - y^2}{0^2 + y^2} = \frac{-y^2}{y^2} = -1$$

we deduce $$\lim_{(x,y)\xrightarrow{x=0}(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} h(x) = -1.$$

The limit being different in two distinct directions, we deduce that

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ *does not exist.*

<u>iii) on the equation parabola $y = x^2$ we have:</u>

$$k(x) := f(x, x^2) = \frac{x^2 - x^4}{x^2 + x^4} = \frac{1 - x^2}{1 + x^2}$$

we deduce $$\lim_{(x,y)\xrightarrow{y=x^2}(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} k(x) = +1.$$

<u>**Remark:**</u> we have the same limit on two directions ($y = 0$ and $y = x^2$) but that does not mean that the limit exists (**why?**)

> ***<u>Proposition (Uniqueness of the limit):</u>***
>
> *If a sequence is convergent, its* ***limit is unique****.*

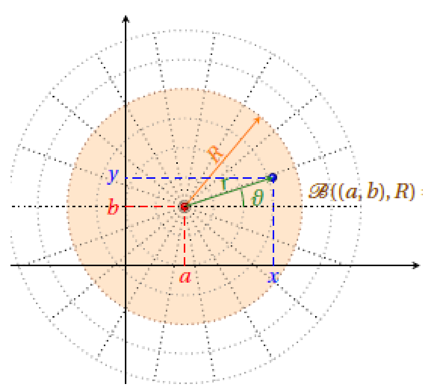
2.1. Calculation of limits in $\mathbb{R}^2$

When $n = 2$, it is often useful to switch to ***polar coordinates***

$$x = a + r\cos\theta \qquad y = b + r\sin\theta$$

to **reduce the calculation** of the limit of a function of two variables x and y to the limit on a **single variable r**.

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{\substack{r\to 0\\ \forall\theta}} f(a + r\cos(\theta), b + r\sin(\theta))$$



We can then use the following sufficient conditions:

> ***Proposition: (sufficient conditions)***
> ***1)*** *If there exists* $L \in \mathbb{R}$*and a function* $s : r \to s(r)$ *such that in the neighborhood of* (a, b)*we have*
>
> $$|f(a + r\cos(\theta), b + r\sin(\theta)) - L| \le s(r) \underset{r \to 0}{\longrightarrow} 0$$
>
> *then* $$\lim_{(x,y)\to(a,b)} f(x, y) = L$$
>
> ***2)*** *If there exists* $L \in \mathbb{R}$ *and a function* $m : r \to m(r)$ *such that in the neighborhood of* (a, b) *we have*
>
> $$|f(a + r\cos(\theta), b + r\sin(\theta)) - L| \ge m(r) \underset{r \to 0}{\longrightarrow} +\infty$$
>
> *then* $\lim_{(x,y)\to(a,b)} f(x, y)$ *does not exist.*

Exercise

Show using polar coordinates that $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$does not exist.

Correction

Let's put it down $x = 0 + r\cos\theta = r\cos\theta$ and $y = 0 + r\sin\theta = r\sin\theta$, we have

$$\frac{x^2 - y^2}{x^2 + y^2} = \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \frac{\cos^2\theta - \sin^2\theta}{1} = \cos(2\theta)$$

consequently $$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{r\to 0} \cos(2\theta) = \cos(2\theta)$$

As θ is arbitrary we can have any values as limit.

So $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$does not exist.

Exercise

Do these limits exist: 1) $\lim_{(x,y)\to(1,1)} \frac{1}{x - y}$ 2) $\lim_{(x,y)\to(1,0)} \frac{y^3}{(x - 1)^2 + y^2}$.

Correction

1) Put $f(x,y)=\dfrac{1}{x-y}$.

$$\lim_{y\to 1^+} k(y) := f(1,y) = \lim_{y\to 1^+} \frac{1}{1-y} = -\infty \quad \text{and} \quad \lim_{x\to 1^-} h(x) := f(x,1) = \lim_{x\to 1^+} \frac{1}{x-1} = +\infty.$$

The limit does not exist.

2) Put $x = 1 + r\cos\theta$ and $y = r\sin\theta$, we have

$$\frac{y^3}{(x-1)^2+y^2} = \frac{r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \frac{r\ \sin\theta}{1} = r\ \sin\theta,$$

then $\displaystyle\lim_{(x,y)\to(1,0)} \frac{y^3}{(x-1)^2+y^2} = \lim_{r\to 0} r\ \sin(\theta) = 0.$

Exercise

Calculate the limit if it exists: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x\ \ln(1+x^3)}{y\,(x^2+y^2)}.$

Correction

Put $f(x,y)=\dfrac{x\ \ln(1+x^3)}{y\,(x^2+y^2)}$, because of the logarithm, let us approach the origin in two different ways, a straight line $(x, y = ax)$and a parabola $(x, y = x^2)$:

$$\lim_{y\to 1^+} k(x) := f(x,ax) = \lim_{x\to 0} \frac{x\ \ln(1+x^3)}{ax\,(x^2+a^2x^2)} = \lim_{x\to 0} \frac{x}{a\,(1+a^2)} \times \frac{\ln(1+x^3)}{x^3} = 0\times 1 = 0$$

$$\lim_{y\to 1^+} k(x) := f(x,x^2) = \lim_{x\to 0} \frac{x\ \ln(1+x^3)}{x^2\,(x^2+x^4)} = \lim_{x\to 0} \frac{1}{1+x^2} \times \frac{\ln(1+x^3)}{x^3} = 1.$$

We deduce that the limit does not exist.

Exercise

Let *f be* the function defined by $f(x,y)=\dfrac{6x^2y}{x^2+y^2}$. Show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$:

1) according to the definition (use the Euclidean norm),

2) by comparisons, **3)** using polar coordinates.

Correction

1) $\forall \varepsilon > 0$, we have to find $r > 0$ such that $\sqrt{x^2+y^2} \leq r \implies |f(x,y)-0| < \varepsilon$.
We have

$$|f(x,y)| = \frac{6x^2\,|y|}{x^2+y^2} \leq \frac{6x^2\,|y|}{x^2} = 6\,|y| \leq 6\,\sqrt{x^2+y^2} \leq 6r,$$

then if $\varepsilon > 0$ is given, it suffices to take a number $r > 0$such that $6r < \varepsilon$. (for example to $\varepsilon = 0.000\,1$ take $r = 0.000\,01$ and to $\varepsilon = 0.000\,000\,6$ take $r = 0.000\,000\,1$ ...)

2) For everything $(x,y) \neq (0,0)$ we have

$0 \leq |f(x,y)| = \dfrac{6x^2\,|y|}{x^2+y^2} \leq \dfrac{6x^2\,|y|}{x^2} = 6\,|y|$, then

$$0 \leq \lim_{(x,y)\to(0,0)} |f(x,y)| \leq \lim_{(x,y)\to(0,0)} 6\,|y| = 0.$$

Consequently $$\lim_{(x,y)\to(0,0)} |f(x,y)| = \lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

3) Let us put $x = r\cos\theta$ and $y = r\sin\theta$, we have $f(x,y) = \dfrac{6x^2y}{x^2+y^2}$

$$f(x,y) = \frac{6x^2y}{x^2+y^2} = \frac{6r^3\,\cos^2\theta\,\sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \frac{6r\,\cos^2\theta\sin\theta}{1} \xrightarrow[r\to 0]{} 0$$

So $\lim_{(x,y)\to(0,0)} |f(x,y)| = \lim_{(x,y)\to(0,0)} f(x,y) = 0.$

2.2. Continuity

Definition:

1) *$f\ :\ \mathbb{R}^n \to \mathbb{R}$ is continuous at $A \in \mathbb{R}^n$ if*

$$\lim_{X\to A} f(X) = f(A).$$

2) *$f\ :\ \mathbb{R}^2 \to \mathbb{R}$ is continuous on domain $D \subset \mathbb{R}^2$ if it is continuous at every point of D.*

Proposition (Properties):

- ✓ *Continuous functions of several variables enjoy the **same properties as continuous functions of a single variable**,*

- ✓ ***Elementary functions** such as polynomials, exponential, logarithmic and trigonometric functions are **continuous** in their respective domains of definition.*
- ✓ *The **composite** (sum, product, quotient etc...) **of continuous** functions is a **continuous** function.*

Example :

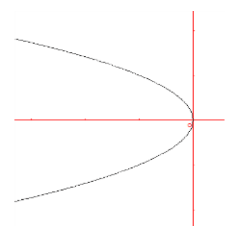
1) $f(x,y) = x^2 + y^2 - xy + y$ is continuous in $\mathbb{R}^2$ (second degree polynomial in two variables).

2) $f(x,y,z) = e^y + xy^2 - z$ is continuous in $\mathbb{R}^3$ (sum of an exponential and a polynomial).

3) $f(x,y) = ln(x + y^2) - 3$.

$$(x,y) \in D_f \iff x + y^2 > 0 \iff x < 0 \; et \; |y| > \sqrt{-x}.$$

f is continuous on D (exterior of the parabola opposite in the left half-plane) as the sum of the logarithm of a polynomial (compound function) and a constant.



Exercise

the multivariable f function be defined on $\mathbb{R}^2$ by

$$f(x) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2} & \text{si } (x,y) \neq (0,0) \\ 0 & \text{si } (x,y) = (0,0) \end{cases}$$

f is it continuous on $\mathbb{R}^2$?

Correction

For $(x,y) \neq (0,0)$ the function $(x,y) \to \dfrac{x^3 + y^3}{x^2 + y^2}$ is rational therefore it is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.

To study the continuity at the point $(0,0)$, let us put $x = r\cos\theta$ and $y = r\sin\theta$, then for $(x,y) \neq (0,0)$ we'll have

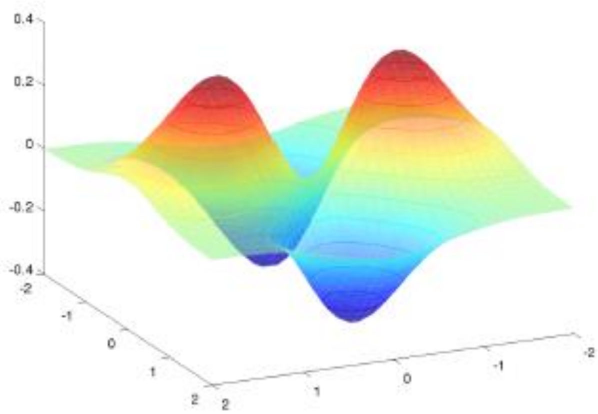
$$f(x,y) = \frac{x^3 + y^3}{x^2 + y^2} = \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = r\,\frac{\cos^3\theta + \sin^3\theta}{1} \xrightarrow[r \to 0]{} 0;$$

so $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$where f continues at (0,0).
Function f is- continues on $\mathbb{R}^2$.

2.3. Theorem extreme values

Theorem : (Extreme Values)

Let $D \subset \mathbb{R}^n$ **compact** ***(i.e. closed** and **bounded)** . If a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is **continuous** then it **admits** a maximum and a minimum ("**extreme values**") on $D \subset \mathbb{R}^n$; i.e. it exists $X_m, X_M \in D$ such that* $(\forall\, X \in D) : f(X_m) \leq f(X) \leq f(X_M)$.



3 DERIVATION AND DIFFERENTIABILITY IN $\mathbb{R}^n$

3.1. Directional derivatives

The unique derivative of a function $f : \mathbb{R} \to \mathbb{R}$, when it exists, is linked to variations in the function as ***the variable travels along the x-axis*** .
It is given by

$$\lim_{h\to 0} \frac{\Delta y}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Note that the real axis offers only **one possible direction** of movement (***horizontal***).

For a function with **two variables** $f : \mathbb{R}^2 \to \mathbb{R}$, whose graph is a surface of $\mathbb{R}^3$, the situation is very different. In fact, in the plane $\mathbb{R}^2$ there is **an infinity of possible directions**.

It may be **interesting** to study how a function $f:\mathbb{R}^2\to\mathbb{R}$ involves when the variable follows one or the other direction of the plan. We can then speak of a **directional derivative** of the function f, which is given by the limit of the rate of increase when $f$its argument (*x* , *y*) vary in a **fixed direction**:

$$\lim_{h\to 0}\frac{\Delta z}{h}=\lim_{h\to 0}\frac{f\left((x,y)+h\,V\right)-f(x,y)}{h}$$

where $V\in\mathbb{R}^2$ is a given direction.

N.B.: for reasons of simplification we will now treat the case **n=2** (*possibly n=3*). The general case is done in the same way.

In this case the previous limit is written when $V=(a,b)$:

$$\lim_{h\to 0}\frac{f(x+h\,a,y+h\,b)-f(x,y)}{h}.$$

3.2. First order partial derivatives and gradient

The set of variables (*the plane* $\mathbb{R}^2$*in our case*) being provided with **two reference directions** ($V_1=(1,0)$ *for **x axis** and* $V_2=(0,1)$ *for **y axis***) gives special interest to the derivatives in these directions which will be called **partial derivatives**.

3.2.1. Partial derivation

Definition: (Partial derivatives)

Let us be a function with multiple variables and real values $f:\mathbb{R}^2\to\mathbb{R}$ *defined on an **open domain*** $D\subset\mathbb{R}^2$ *and a point* $(x_0,y_0)\in D$.

- ✓ *The* **partial derivatives** *of* f *at* (x_0,y_0) *is the **derivatives of the partial functions*** $f_{y_0}:x\to f(x,y_0)$ *and* $f_{x_0}:y\to f(x_0,y)$ *i.e.*

1) **partial derivative of f with respect to x** *at the point* (x_0,y_0)*:*

$$\frac{\partial f}{\partial x}(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}.$$

2) **partial derivative of f with respect to y** *at the point* (x_0,y_0)*:*

$$\frac{\partial f}{\partial y}(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}.$$

✓ *If **all** first **partial derivatives of** f **exist**, we say that f is* **differentiable**.

Notation : the partial derivative $\frac{\partial f}{\partial x}$ (respectively $\frac{\partial f}{\partial y}$) is also noted $\partial_x f$ (respectively $\partial_y f$) or f'_x (respectively f'_y).

Remark :

In practice, to calculate the partial derivative $\frac{\partial f}{\partial x}$ (resp. $\frac{\partial f}{\partial y}$), we derive f *such* as it is a function of the single variable x $(resp.\ y)$ with the other variable, y $(resp.\ x)$ considered as constant.

Example :

Let the function be $(x, y) \to f(x, y) = 4 - x^2 - 2y^2$. We have

$$f'_x(x, y) = \frac{\partial f}{\partial x}(x, y) = -2x \qquad \text{and} \qquad f'_y(x, y) = \frac{\partial f}{\partial y}(x, y) = -4y.$$

Exercise

The annual production of wheat *B* depends on the average temperature T *and* the average precipitation *R.* Scientists estimate that the average temperature is increasing by $0.15\ °C/an$ and precipitation is decreasing by $0.1\ cm/an$. They also think that for the current level of production $B(T, R)$ we have $\partial_T B = -2$and $\partial_R B = +8$.

1) Write in terms of ratio the variations of temperature and precipitation?
2) What do these partial derivatives mean?

Exercise
Correction

1) The average temperature is increasing at a rate of $0.15\ °C/an$: $\frac{dT}{dt} = +0.15$.

Precipitation decreases by $0.1\ cm/an$: $\frac{dR}{dt} = -0.1$

2) . $\partial_T B = -2$: An increase in average temperature (while keeping annual precipitation constant) results in a ***decrease in wheat production*** at current production levels.

$\partial_R B = +8$: an increase in annual rainfall (while keeping the average temperature constant) causes an ***increase in wheat production***.

3.2.2. Properties of partial derivation

> **Properties: (of the partial derivation)**
>
> *Partial derivatives have the **same properties as derivatives of functions of a single variable**. Especially:*
>
> ✓ ***Elementary functions** such as polynomials, rational and irrational functions, exponential, logarithmic and trigonometric functions **are differentiable** in their respective domains.*
>
> ✓ *The sum, product, quotient, etc. of differentiable functions is a differentiable function.*
>
> ✓ *The derivation **rules are similar** to the derivation rules with a single variable, (except that relating to the derivation of compound functions which are less simple to define).*

Example

1) Let f defined by $f(x,y) = 3x^2 + xy - 2y^2$. f is continuous and derivable (polynomial function):

y considered constant we obtain: $f'_x(x,y) = \dfrac{\partial f}{\partial x}(x,y) = 6x + y$.

x considered constant we obtain: $f'_y(x,y) = \dfrac{\partial f}{\partial y}(x,y) = x - 4y$.

2) Let f defined by $f(x,y,z) = 5xz\ln(1+7y)$. f is continuous and derivable (compound of polynomials and logarithm):

y and z considered as constants gives:

$$\partial_x f = f'_x(x,y) = \frac{\partial f}{\partial x}(x,y) = 5z\ln(1+7y).$$

x and z considered as constants gives:

$$\partial_y f = f'_y(x,y) = \frac{\partial f}{\partial y}(x,y) = 5xz\frac{7}{1+7y}.$$

x and y considered as constants gives:

$$\partial_z f = f'_z(x, y) = \frac{\partial f}{\partial z}(x, y) = 5x \ln(1 + 7y).$$

3.2.3. Gradient

Definition: (Gradient)

The **gradient** *of the function $f : \mathbb{R}^n \to \mathbb{R}$ evaluated at the point $A = (a_1, a_2, .., a_n)$, noted $\nabla f(A)$ (**reads nabla f at point A**) or again $grad\, f(A)$, is the **vector whose components are** the **first partial derivatives** of f :*

$$\nabla f(A) = grad\, f(A) = \begin{pmatrix} \partial_{x_1} f \\ \partial_{x_2} f \\ \ddots \\ \partial_{x_n} f \end{pmatrix} = \left(\partial_{x_1} f\ ,\ \partial_{x_2} f\ ,\ \dots\ ,\ \partial_{x_n}\right)^T.$$

It is orthogonal to the level curve of f passing by A.

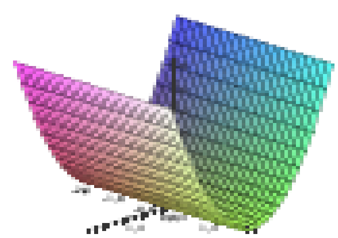
Example

1) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 + y$.

The gradient of f is the vector $\nabla f(x, y) = grad\, f(x, y) = \left(2x\ ,\ 1\right)^T$.

2)Consider $g(x, y) = x^2$ then

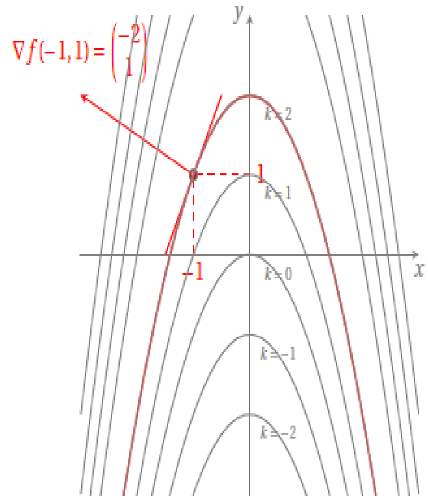
$$\begin{cases} g(x, b) = x^2 & parbola \\ g(a, y) = cste & horizontal\ line \end{cases}$$



We obtain the representative graphic C_g of g

One can deduce the curve of f resulting on the inclination $+45°$ because of value along y-axis

$$\begin{cases} f(x,b) = x^2 & parbol \\ f(a,y) = y & oblique\ line \end{cases}.$$



The level curves of the function are given by

$$f(x,y) = x^2 + y = k, \quad k \in \mathbb{R}.$$

It is a family of parabolas of equations

$$y = -x^2 + k \ , \ k \in \mathbb{R}$$

The gradient is orthogonal to the level curve which passes through the point (x, y).

In the figure above we consider the point $(-1,1)$ in the contour curve $k = 2$ which has the equation $y = -x^2 + 2$. We have in this point

$$\nabla f(-1,1) = grad\, f(-1,1) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

The line tangent to this curve at the point $(-1,1)$ has the equation $y = 2x + 3$ which is orthogonal to $grad\, f(x,y)$. Indeed, the direction vector of the tangent is $V = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ hence the scalar product $\nabla f(x,y)\ .\ V = (-2)(1) + (1)(2) = 0$, which proves that the two vectors are orthogonal.

Exercise

Calculate the partial derivatives of order 1 of the following functions and write the gradient:

1) $f(x,y) = y^5 - 3xy$, 2) $f(x,y) = x^2 + 3xy^2 - 6y^5$, 3) $f(x,y) = x\cos(e^{xy})$,
4) $f(x,y) = \dfrac{x}{y}$, 5) $f(x,t) = e^{-t}\cos(\pi\, x)$.

Correction

1) For $(x,y) \in \mathbb{R}^2$ we have:

$$\partial_x f(x,y) = -3y \quad \text{and} \quad \partial_y f(x,y) = 5y^4 - 3x.$$

$$\nabla f(x,y) = grad\, f(x,y) = \begin{pmatrix} -3y \\ 5y^4 - 3x \end{pmatrix}$$

2) For $(x,y) \in \mathbb{R}^2$ we have:

$\partial_x f(x,y) = 2x + 3y^2$ and $\partial_y f(x,y) = 6xy - 30y^4$.

$$\nabla f(x,y) = grad\, f(x,y) = \begin{pmatrix} 2x + 3y^2 \\ 6xy - 30y^4 \end{pmatrix}$$

3) For $(x,y) \in \mathbb{R}^2$ we have:

$\partial_x f(x,y) = \cos(e^{xy}) - xye^{xy}\sin(e^{xy})$ and $\partial_y f(x,y) = -x^2e^{xy}\cos(e^{xy})$.

$$\nabla f(x,y) = grad\, f(x,y) = \begin{pmatrix} \cos(e^{xy}) - xye^{xy}\sin(e^{xy}) \\ -x^2e^{xy}\cos(e^{xy}) \end{pmatrix}$$

4) For $(x,y) \in \mathbb{R}^2 \setminus \{(x,0) : x \in \mathbb{R}\}$ we have:

$\partial_x f(x,y) = \frac{1}{y}$ and $\partial_y f(x,y) = -\frac{x}{y^2}$.

$$\nabla f(x,y) = grad\, f(x,y) = \begin{pmatrix} \frac{1}{y} \\ -\frac{x}{y^2} \end{pmatrix}$$

5) For $(x,y) \in \mathbb{R}^2 \setminus \{(x,0) : x \in \mathbb{R}\}$ we have:

$\partial_x f(x,t) = -\pi\, e^{-t}\sin(x)$ and $f(x,t) = -e^{-t}\cos(\pi\, x)$

$$\nabla f(x,y) = grad\, f(x,y) = \begin{pmatrix} -\pi\, e^{-t}\sin(x) \\ -e^{-t}\cos(\pi\, x) \end{pmatrix}.$$

3.2.4. Derivability and continuity

Unlike $\mathbb{R}$, in $\mathbb{R}^n$, $n \geq 2$ the derivative existence (derivability) is independent of continuity.

1) $f : (x,y) \to f(x,y) = |x| + |y|$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$ is **continuous and not derivable** at $(0,0)$.

2) $f(x,y) = \dfrac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$ is **derivable and not continuous** at $(0,0)$.

3) $f(x,y) = \dfrac{y}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$ is **neither derivable nor continues** in $(0,0)$.

3.2.5. Class function C^1

Definition: (class C^1)

*If a function$f : D \subset \mathbb{R}^n \to \mathbb{R}$ is **derivable** and its **partial derivative functions** $\partial_{x_i} f$, $i = 1, .., n$ **are continuous** on D, we say that f is **of class C^1** on D and we denote $f \in C^1(D)$.*

3.3. Compound functions

Recall that the compound function of f and g is defined as follows:

$$\forall t \ : \ (f \circ g)(t) := f[g(t)].$$

Similarly for a function with multiple variables $(x, y) \to f(x, y)$, the variables x and y can be functions with one variables $t \in \mathbb{R}$ (or many variables).

3.3.1. Case of a single variable

compound function :

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \to f(x, y)$ where the variables x and y are functions with variable $t \in \mathbb{R}$. We pose

$$f(x(t), y(t)) = g(t).$$

If the partial derivatives $\partial_x f$ and $\partial_x f$ of f exist and as functions (from $\mathbb{R}$ to $\mathbb{R}$) $t \to x(t)$ and $t \to y(t)$ are derivable, then the function $g : t \to g(t) := f(x(t), y(t))$ is derivable and we have

$$g'(t) = \frac{\partial f}{\partial x}\big(x(t), y(t)\big) \times x'(t) + \frac{\partial f}{\partial y}\big(x(t), y(t)\big) \times y'(t).$$

N.B.: it might be simpler to remember this using differential notations:

$$g'(t) = \frac{dg}{dt}(t) = \frac{df}{dt}\big(x(t), y(t)\big) = \frac{\partial f}{\partial x}\big(x(t), y(t)\big) \times \frac{dx}{dt}(t) + \frac{\partial f}{\partial y}\big(x(t), y(t)\big) \times \frac{dy}{dt}(t).$$

Example

Annual wheat production $B(T, R)$ is a function of the average temperature T and the average precipitation R. Scientists estimate that $\partial_T B = -2$and

$\partial_R B = +8$. Knowing that the average temperature is increasing by $0.15\ °C/an$ and precipitation decreases by $0.1\ cm/an$; estimate the current rate of change in wheat production *dB/dt* .

Correction

The current rate of change in wheat production is *dB/dt*

$$\frac{dB}{dt}\big(T(t),R(t)\big) = \frac{\partial B}{\partial T}\big(T,R\big) \times \frac{dT}{dt}(t) + \frac{\partial B}{\partial R}\big(T,R\big) \times \frac{dR}{dt}(t).$$

As the average temperature increases at a rate of $0.15\ °C/an$ we have $\frac{dT}{dt} = 0.15$; and as precipitation decreases at the rate of $0.1\ cm/an$: $\frac{dR}{dt} = -0.1$; from where

$$\begin{aligned}\frac{dB}{dt}\big(T(t),R(t)\big) &= \partial_T B\big(T,R\big) \times (0.15) + \partial_R B\big(T,R\big) \times (-0.1) \\ &= (-2) \times (0.15) + (8) \times (-0.1) \ = -1.1\end{aligned}.$$

Exercise

Calculate $g'(t)$ in the following cases:

1) $g(t) = f(x(t),y(t))\ ,\ f(x,y) = x^2 + y^2 + xy\ ,\ x(t) = \sin t\ ,\ y(t) = e^t$.

2) $g(t) = f(x(t),y(t))\ ,\ f(x,y) = \cos(x+4y)\ ,\ x(t) = 5t^4\ ,\ y(t) = \frac{1}{t}$.

3) $g(t) = f(x(t),y(t),z(t))\ ,\ f(x,y,z) = xe^{y/z}\ ,\ x(t) = t^2\ ,\ y(t) = 1-t\ ,\ z(t) = 1+2t$.

4) $g(t) = f(x(t),y(t),z(t))\ ,\ f(x,y,z) = \ln(\sqrt{x^2+y^2+z^2})\ ,\ x(t) = \sin t\ ,\ y(t) = \cos t\ ,\ z(t) = \tan t$.

Correction

1)

$$\begin{aligned}g'(t) &= f'_x(x(t),y(t)) \times x'(t) + f'_y(x(t),y(t)) \times y'(t) \\ &= (2x+y) \times \cos t + (2y+x) \times e^t \\ &= (2\sin t + e^t)\ \cos t + (2e^t + \sin t)\, e^t\ .\end{aligned}$$

2)

$$\begin{aligned}g'(t) &= f'_x(x(t),y(t)) \times x'(t) + f'_y(x(t),y(t)) \times y'(t) \\ &= \cos(x+4y) \times 20t^3 + 4\cos(x+4y) \times \frac{-1}{t^2} \\ &= 20t^3\ \cos(x+4y) - \frac{4}{t^2}\ \cos(x+4y)\ .\end{aligned}$$

3)

$$\begin{aligned} g'(t) &= f'_x(x(t),y(t),z(t)) \times x'(t) + f'_y(x(t),y(t),z(t)) \times y'(t) + f'_z(x(t),y(t),z(t)) \times z'(t) \\ &= e^{(1-t)/(1+2t)} \times 2t + \frac{x}{z}e^{y/z} \times (-1) + \frac{-xy}{z^2}e^{y/z} \times 2 \\ &= \left[2t - \frac{t^2}{1+2t} - 2\frac{t^2(1-t)}{(1+2t)^2}\right] e^{(1-t)/(1+2t)} = \frac{8t^3+5t^2+2t}{(1+2t)^2}\, e^{(1-t)/(1+2t)}\ . \end{aligned}$$

4)

$$\begin{aligned} g'(t) &= f'_x(x(t),y(t),z(t)) \times x'(t) + f'_y(x(t),y(t),z(t)) \times y'(t) + f'_z(x(t),y(t),z(t)) \times z'(t) \\ &= \frac{1}{2}\frac{2x}{x^2+y^2+z^2} \times \cos t + \frac{1}{2}\frac{2y}{x^2+y^2+z^2} \times (-\sin t) + \frac{1}{2}\frac{2z}{x^2+y^2+z^2} \times \frac{1}{\cos^2 t} \\ &= \frac{1}{2}\frac{2\sin t \times \cos t}{1+\tan^2 t} - \frac{1}{2}\frac{2\cos t \times \sin t}{1+\tan^2 t} + \frac{1}{2}\frac{2\tan t}{1+\tan^2 t} \times \frac{1}{\cos^2 t} \\ &= \frac{1}{2}\ \frac{2\tan t}{1+\tan^2 t} \times \frac{1}{\cos^2 t} = \frac{1}{2}\ \frac{2\sin t}{\cos t}\ \frac{\cos^2 t}{\cos^2 t+\sin^2 t} \times \frac{1}{\cos^2 t} = \frac{\sin t}{\cos t}\ . \end{aligned}$$

3.3.2. Case of two variables

Compound function :

Consider the function $f:\mathbb{R}^2 \to \mathbb{R}$, $(x,y) \to f(x,y)$*where the variables* x *and* y *are functions with two variables* $u,v \in \mathbb{R}$*. We pose*

$$f(x(u,v),y(u,v)) = g(u,v).$$

If the partial derivatives $\partial_x f$ *and* $\partial_x f$ *of* f *exist, the partial derivatives* $\partial_u x$ *and* $\partial_v x$ *of* x*and the partial derivatives* $\partial_u y$ *and* $\partial_v y$ *of* y *exist, then the function*

$$g: t \to g(u,v) := f(x(u,v),y(u,v))$$

is derivable (i.e. admits partial derivatives) and we have

$$\begin{aligned} \frac{\partial g}{\partial u}(u,v) &= \frac{\partial f}{\partial u}\big(x(u,v),y(u,v)\big) \\ &= \frac{\partial f}{\partial x}\big(x(t),y(t)\big) \times \frac{\partial x}{\partial u}(u,v) + \frac{\partial f}{\partial y}\big(x(t),y(t)\big) \times \frac{\partial y}{\partial u}(u,v) \end{aligned}.$$

$$\begin{aligned} \frac{\partial g}{\partial v}(u,v) &= \frac{\partial f}{\partial u}\big(x(u,v),y(u,v)\big) \\ &= \frac{\partial f}{\partial x}\big(x(t),y(t)\big) \times \frac{\partial x}{\partial v}(u,v) + \frac{\partial f}{\partial y}\big(x(t),y(t)\big) \times \frac{\partial y}{\partial v}(u,v) \end{aligned}.$$

Exercise

The functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ being given, calculate $\partial_x\psi(x,y)$ and $\partial_y\psi(x,y)$ in the following cases:

1) $\psi(x,y)=f(x)+g(y)$, 2) $\psi(x,y)=f(x)\,g(y)$, 3),

4) $\psi(x,y)=f(x+2y)$, 5) $\psi(x,y)=f(x\,y)$, 6) $\psi(x,y)=f\left(\frac{x}{y}\right)$.

Correction

1) $\partial_x\psi(x,y)=f'(x)$ and $\partial_y\psi(x,y)=g'(x)$.

2) $\partial_x\psi(x,y)=f'(x)\,g(y)$ and $\partial_y\psi(x,y)=f(x)\,g'(x)$.

3) $\partial_x\psi(x,y)=\dfrac{f'(x)}{g(y)}$ and $\partial_x\psi(x,y)=-\dfrac{f(x)\;g'(x)}{g^2(y)}$.

4) $\partial_x\psi(x,y)=f'(x+2y)$ and $\partial_y\psi(x,y)=2f'(x+2y)$

5) $\partial_x\psi(x,y)=y\,f'(x\,y)$ and $\partial_y\psi(x,y)=x\,f'(x\,y)$,

6) $\partial_x\psi(x,y)=\frac{1}{y}\,f'\left(\frac{x}{y}\right)$ and $\partial_y\psi(x,y)=-\frac{x}{y^2}\,f'\left(\frac{x}{y}\right)$.

3.4. Differentiability

Differentiability at a point **corresponds to the existence of a linear approximation** of the function at that point.

For a function $f:\mathbb{R}\to\mathbb{R}$ (with a single variable), geometrically this corresponds to the existence of a ***line tangent*** to the graph in the neighborhood of the point $(x_0,f(x_0))$. We know that there is **equivalence** between ***differentiability*** and **derivability**.

In the case of functions of several, the equivalence disappears between (derivability) the existence of partial derivatives $\dfrac{\partial f}{\partial x}(x_0,y_0,\ldots)$, $\dfrac{\partial f}{\partial y}(x_0,y_0,\ldots)$, etc... and existence of a tangent plane.

3.4.1. Differentiable function

Definition: (differentiable function)

Let f be a function with multiple variables and real values defined on an ***open set*** *$D \subset \mathbb{R}^2$ and let $(x_0, y_0) \in D$.*

We say that f is **differentiable** *at (x_0, y_0) if there exist two constants $A, B \in \mathbb{R}$ such that*

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = A\,h + B\,k + \circ(\|(h,k)\|).$$

N.B.: remember that $\circ(\|(h,k)\|) = \|(h,k)\|\,\epsilon(h,k)$ with $\epsilon(h,k) \xrightarrow[h,k\to 0]{} 0$ where $\|(h,k)\| = \sqrt{h^2+k^2}$ (we can use another equivalent norm like $\|(h,k)\| = \max(|h|,|k|)$).

*The application $(h,k) \to A\,h + B\,k$ is linear and it represents an approximation (***linear approximation*** or* ***approximation of first order****) of f in the vicinity of (x_0, y_0).*

Remark :

If f is derivable at (x_0, y_0) **i.e.** partial derivatives $\partial_x f(x_0, y_0)$ and $\partial_y f(x_0, y_0)$ exist, then the linear map

$$(h,k) \to \partial_x f(x_0, y_0)\,h + \partial_y f(x_0, y_0)\,k$$

is ***candidate to be approximation*** (*linear or of first order*) of f in the neighborhood of (x_0, y_0). To confirm, just check if

$$\frac{f(x_0+h, y_0+k) - f(x_0,y_0) - \partial_x f(x_0,y_0)\,h - \partial_y f(x_0,y_0)\,k}{\sqrt{h^2+k^2}} \xrightarrow[(h,k)\to(0,0)]{} 0$$

Exercise

Using the definition, verify that the $f : \mathbb{R}^2 \to \mathbb{R}$ following functions are differentiable at the point (x_0, y_0):

1) $f(x,y) = xy - 3x^2$, $(x_0,y_0) = (1;2)$, 2), $f(x,y) = xy - 3y^2$, $(x_0,y_0) = (2;1)$

3) $f(x,y) = y\sqrt{x}$, $(x_0,y_0) = (4;1)$, 4), $f(x,y) = |y|\,\ln(1+x)$, $(x_0,y_0) = (0;0)$.

Correction

1) We have $f(x,y) = xy - 3x^2$, $(x_0, y_0) = (1;2)$, $f(1,2) = -1$,
$\partial_x f(x,y) = y - 6x \implies \partial_x f(1,2) = -4$ and $\partial_y f(x,y) = x \implies \partial_y f(1,2) = 1$ then

$$\begin{aligned}
E(h,k) &= \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - \partial_x f(x_0,y_0)\,h - \partial_y f(x_0,y_0)\,k}{\sqrt{h^2+k^2}} \\
&= \frac{(1+h)(2+k) - 3(1+h)^2 - (-1) - (-4)\,h - 1\,k}{\sqrt{h^2+k^2}} \\
&= \frac{2+2h+k+hk-3-6h-3h^2+1+4h-k}{\sqrt{h^2+k^2}} \\
&= \frac{-3h^2+hk}{\sqrt{h^2+k^2}} = \frac{-3r^2\cos^2 t + r^2 \cos t \sin t}{r\,\sqrt{\cos^2 t + \sin^2 t}}\ ;
\end{aligned}$$

we deduce $|E(h,k)| \leq \dfrac{4r^2}{r} = 4r \underset{r\to 0}{\longrightarrow} 0$. Therefore $(x,y) \to f(x,y) = xy - 3x^2$ is differentiable at $(x_0, y_0) = (1;2)$.

2) We have $f(x,y) = xy - 3y^2$, $(x_0, y_0) = (2;1)$, $f(2,1) = -1$,
$\partial_x f(x,y) = y \implies \partial_x f(2,1) = 1$ and $\partial_y f(x,y) = x - 6y \implies \partial_y f(2,1) = -4$ then

$$\begin{aligned}
E(h,k) &= \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - \partial_x f(x_0,y_0)\,h - \partial_y f(x_0,y_0)\,k}{\sqrt{h^2+k^2}} \\
&= \frac{(2+h)(1+k) - 3(1+k)^2 - (-1) - 1\,h - (-4)\,k}{\sqrt{h^2+k^2}} \\
&= \frac{2+h+2k+hk-3-6k-3k^2+1-h+4k}{\sqrt{h^2+k^2}} \\
&= \frac{-3k^2+hk}{\sqrt{h^2+k^2}} = \frac{-3r^2\sin^2 t + r^2 \cos t \sin t}{r\,\sqrt{\cos^2 t + \sin^2 t}}\ ;
\end{aligned}$$

we deduce $|E(h,k)| \leq \dfrac{4r^2}{r} = 4r \underset{r\to 0}{\longrightarrow} 0$; therefore $(x,y) \to f(x,y) = xy - 3y^2$ is differentiable at $(x_0, y_0) = (2;1)$.

3) We have $f(x,y) = y\sqrt{x}$, $(x_0, y_0) = (4;1)$, $f(4,1) = 2$,
$\partial_x f(x,y) = \dfrac{y}{2\sqrt{x}} \implies \partial_x f(4,1) = \dfrac{1}{4}$ and $\partial_y f(x,y) = \sqrt{x} \implies \partial_y f(2,1) = 2$ then

$$
\begin{aligned}
E(h,k) &= \frac{f(x_0+h,y_0+k)-f(x_0,y_0)-\partial_x f(x_0,y_0)\,h-\partial_y f(x_0,y_0)\,k}{\sqrt{h^2+k^2}}\\
&= \frac{(1+k)\sqrt{4+h}-2-\dfrac{1}{4}\,h-2\,k}{\sqrt{h^2+k^2}}\\
&= \frac{2(1+k)\sqrt{1+\frac{h}{4}}-2-\dfrac{1}{4}\,h-2\,k}{\sqrt{h^2+k^2}}\\
&= \frac{(2+2k)(1+\frac{h}{8}+\circ(h/8))-2-\dfrac{1}{4}\,h-2\,k}{\sqrt{h^2+k^2}}\\
&= \frac{2+\frac{h}{4}+2k+\frac{2kh}{8}+\frac{(2+2k)h}{8}\circ(h)-2-\dfrac{1}{4}\,h-2\,k}{\sqrt{h^2+k^2}}\\
&= \frac{\frac{2kh}{8}+\frac{(2+2k)h}{8}\circ(h)}{\sqrt{h^2+k^2}}\\
&= \frac{2r^2\cos t\sin t+2\cos t\circ(r)+2r^2\cos^2 t\sin t\circ(r)}{8r\sqrt{\cos^2 t+\sin^2 t}}\ ;
\end{aligned}
$$

we deduce $|E(h,k)|\leq \dfrac{r}{4}+\dfrac{\circ(r)}{4r}+\dfrac{r\circ(r)}{4}\ \underset{r\to 0}{\longrightarrow} 0$; therefore $(x,y)\to f(x,y)=y\sqrt{x}$ is differentiable at $(x_0,y_0)=(4;1)$.

NB: we used the equivalence $\sqrt{1+\dfrac{h}{4}}=1+\dfrac{h}{8}+\circ(h/8)$ near 0.

4) We have $f(x,y)=|y|\,\ln(1+x)$, $(x_0,y_0)=(0;0)$ $f(0,0)=0$,

$\partial_x f(x,y)=\dfrac{|y|}{1+x}\implies \partial_x f(0,0)=0$ and $\partial_y f(x,y)=\pm\ln(1+x)\implies \partial_y f(0,0)=0$ then

$$
\begin{aligned}
E(h,k) &= \frac{f(x_0+h,y_0+k)-f(x_0,y_0)-\partial_x f(x_0,y_0)\,h-\partial_y f(x_0,y_0)\,k}{\sqrt{h^2+k^2}}\\
&= \frac{|k|\,\ln(1+h)-0-0\times h-0\times k}{\sqrt{h^2+k^2}}\\
&= \frac{|k|\,(h+\circ(h))}{\sqrt{h^2+k^2}}=\frac{r^2|\sin t|\,\cos t+r|\sin t|\,\cos t\ \circ(r)}{r\sqrt{\cos^2 t+\sin^2 t}}\ ;
\end{aligned}
$$

we deduce $|E(h,k)| \le r + \circ(r) \underset{r\to 0}{\longrightarrow} 0$; therefore $(x,y) \to f(x,y) = |y| \ln(1+x)$ is differentiable at $(x_0, y_0) = (4;1)$.

NB: we used the equivalence $\ln(1+h) = h + \circ(h)$ near 0.

> ***Definition: (the differential)***
> *Let f a function be differentiable at (x_0, y_0). The **linear application** $df(x_0, y_0)$ defined by*
>
> $$df(x_0,y_0) := \frac{\partial f}{\partial x}(x_0,y_0)\ dx + \frac{\partial f}{\partial y}(x_0,y_0)\ dy$$
>
> *is called* **differential function** *of f at (x_0, y_0).*

Example

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = 1 + x^2 - xy$.

f is continuous and differentiable (polynomial function); we have $\frac{\partial f}{\partial x}(x_0,y_0) = 2x_0 - y_0$ and $\frac{\partial f}{\partial x}(x_0,y_0) = -x_0$. The candidate linear map to be the differential is the function $g : (h,k) \to g(h,k) = (2x_0 - y_0)\ h + (-x_0)\ k$.

$$\begin{aligned} f(x_0+h, y_0+k) - f(x_0,y_0) &= 1 + (x_0+h)^2 - (x_0+h)(y_0+k) - 1 - x_0^2 + x_0y_0 \\ &= x_0^2 + 2x_0h + h^2 - x_0y_0 - hy_0 - x_0k - hk - x_0^2 + x_0y_0 \\ &= 2x_0h + h^2 - hy_0 - x_0k - hk \\ &= (2x_0 - y_0)\, h - x_0\, k + (h^2 - hk) \\ &= g(h,k) + (h^2 - hk)\ . \end{aligned}$$

In addition we have $\lim\limits_{h,k\to 0} \frac{h^2 - hk}{\sqrt{h^2+k^2}} = 0$, therefore $(h^2 - hk) = \circ(\|(h,k)\|$.

(x_0, y_0) being arbitrary, we have just shown that f is everywhere differentiable and that its differential is defined by

$$df(x,y) = (2x - y)\, dx - x\, dy.$$

Example

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 + y^2$.
f is continuous and differentiable (polynomial function).

We have $\dfrac{\partial f}{\partial x}(x_0, y_0) = 2x_0$ and $\dfrac{\partial f}{\partial x}(x_0, y_0) = 2y_0$.

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0, y_0) - h\,\partial_x f(x_0, y_0) - k\,\partial_y f(x_0, y_0)}{\sqrt{h^2+k^2}}$$
$$\lim_{(h,k)\to(0,0)} \frac{(x_0+h)^2 + (y_0+k)^2 - x_0^2 - y_0^2 - h\,2x_0 - k\,2y_0}{\sqrt{h^2+k^2}}$$
$$\lim_{(h,k)\to(0,0)} \frac{x_0^2 + 2x_0h + h^2 + y_0^2 + 2y_0k + k^2 - x_0^2 - y_0^2 - h\,2x_0 - k\,2y_0}{\sqrt{h^2+k^2}}$$
$$\lim_{(h,k)\to(0,0)} \frac{h^2+k^2}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \sqrt{h^2+k^2} = 0$$

Consequently, f is everywhere differentiable and has the differential

$$df(x, y) = 2x\,dx + 2y\,dy.$$

3.4.2. Class C^1 implies differentiability

> ***Theorem: ($f \in C^1$ stronger than differentiability)***
> *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined on an **open set** $D \subset \mathbb{R}^2$ and let $(x_0, y_0) \in D$.*
>
> *If f is **of class** C^1 in the neighborhood of (x_0, y_0) (**partial derivatives exist and are continuous**) **then** f is **differentiable** at (x_0, y_0).*

Example

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & \text{si } (x, y) \neq (0, 0) \\ 0 & \text{si } (x, y) = (0, 0) \end{cases}$$

Show that f is of class $C^1(\mathbb{R}^2)$? is it differentiable?

Correction

f *continuous ?*

For $(x,y) \neq (0,0)$ the rational function with non-zero denominator

$(x,y) \to \dfrac{x^2y^2}{x^2+y^2}$ is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.

For $(x,y) = (0,0)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2\,y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^4\cos^2 t\,\sin^2 t}{r^2(\cos^2 t+\sin^2 t)} = \lim_{r\to 0} r^2\cos^2 t\,\sin^2 t = 0.$$

f is continuous at (0,0). So f is everywhere continues.

f *derivable ?*

For $(x,y) \neq (0,0)$ we have

$$(x,y) \to \partial_x f(x,y) = \frac{2x(x^2+y^2)y^2 - x^2(2x)y^2}{(x^2+y^2)^2} = \frac{2xy^4}{(x^2+y^2)^2},$$

$$(x,y) \to \partial_y f(x,y) = \frac{x^2(x^2+y^2)2y - x^2(2y)y^2}{(x^2+y^2)^2} = \frac{2x^4y}{(x^2+y^2)^2}.$$

For $(x,y) = (0,0)$:

$$\partial_x f(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\dfrac{h^2\times 0}{h^2+0^2}}{h} = 0.$$

$$\partial_y f(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\dfrac{0\times k^2}{0+k^2}}{h} = 0.$$

f is differentiable at (0,0). Therefore is everywhere differentiable.

Class $C^1(\mathbb{R}^2)$?

Partial derivative functions $(x,y) \to \partial_x f(x,y) = \dfrac{2xy^4}{(x^2+y^2)^2}$ and

$(x,y) \to \partial_y f(x,y) = \dfrac{2x^4y}{(x^2+y^2)^2}$ are continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$as rational functions with non-zero denominators. $f \in C^1(\mathbb{R}^2 \setminus \{(0,0)\})$.

For $(x,y)=(0,0)$

$$\lim_{(x,y)\to(0,0)} \partial_x f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2x\,y^4}{(x^2+y^2)^2} = \lim_{r\to 0} \frac{2r^5 \cos t\, \sin^4 t}{r^4(\cos^2 t+\sin^2 t)^2} = \lim_{r\to 0} 2r\cos t\, \sin^4 t = 0.$$

$$\lim_{(x,y)\to(0,0)} \partial_y f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2x^4\,y}{(x^2+y^2)^2} = \lim_{r\to 0} \frac{2r^5 \cos^4 t\, \sin t}{r^4(\cos^2 t+\sin^2 t)^2} = \lim_{r\to 0} 2r\cos^4 t\, \sin t = 0.$$

Partial derivatives are continuous at (0,0). We deduce the partial derivative functions $\partial_x f(x,y)$ and $\partial_y f(x,y)$ are everywhere continuous and hence $f \in C^1(\mathbb{R}^2)$.

f differentiable ?

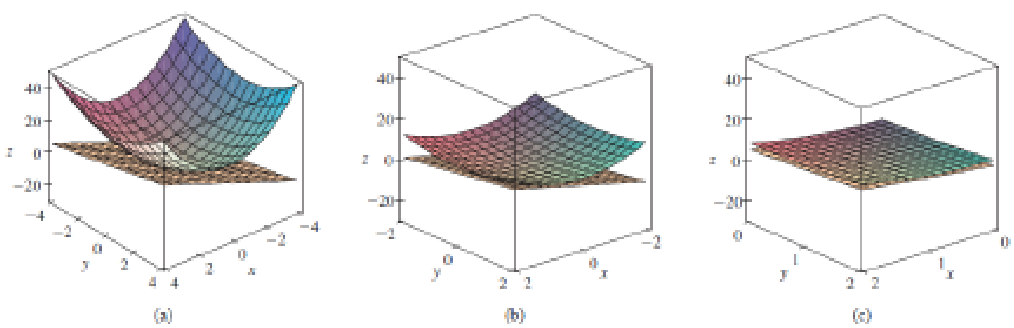
Since $f \in C^1(\mathbb{R}^2)$ then f is differentiable because $classe\ C^1 \implies differentiable$.

N.B.: The converse is false. f can be differentiable without being of class C^1

3.4.3. Tangent plane and linearization

The notion of **differentiability corresponds to the geometric notion of a tangent plane**. Indeed, when f is differentiable at (x_0,y_0), we can, in a neighborhood of (x_0,y_0), approach $f(x_0+h, y_0+k)$ by $f(x_0,y_0)+df(x_0,y_0)(h,k)$.

This corresponds geometrically to approaching the representative surface of f, in the neighborhood of (x_0,y_0), by the plane of equation $z = f(x_0,y_0)+df(x_0,y_0)(h,k)$.



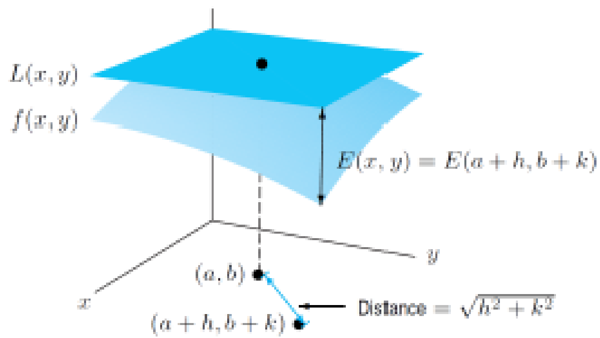
The representative surface of the function $(x,y)\to f(x,y)=2x^2+y$ seems to coincide with its plane tangent to the point $(1,1,f(1,1))$ when we ***zoom towards this point*** .

Definition: (tangent plane)

Let be $f:\mathbb{R}^2\to\mathbb{R}$ *a function defined on an* ***open set*** $D\subset\mathbb{R}^2$ *and differentiable at* $(x_0,y_0)\in D$.

The ***equation*** *of the* **tangent plane**, ***at*** $(\boldsymbol{x_0},\boldsymbol{y_0})$, ***to the graph*** *of the function* f *is*

$$z=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\ (x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)\ (y-y_0).$$



Example

Consider the function $f:\mathbb{R}^2\to\mathbb{R}$defined by

$$f(x,y)=\frac{x+y}{1+x^2+y^2}.$$

1) Determine and represent its contour lines.

2) Calculate its first partial derivatives.

3) Write the equation of the tangent plane to$f$ in (0, 0).

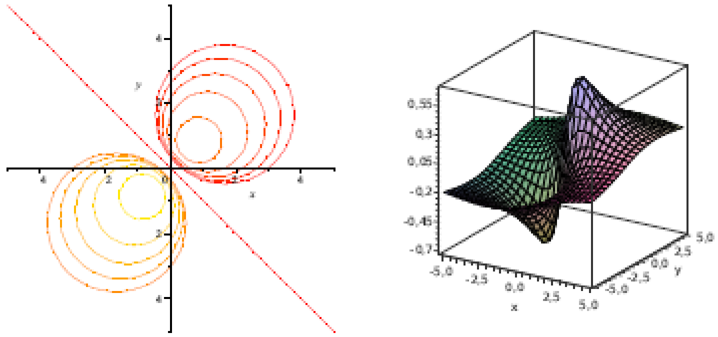
Correction

1) $f(x,y)=\dfrac{x+y}{1+x^2+y^2}=k \iff x+y=0(k=0)\vee 1+x^2+y^2-\dfrac{x+y}{k}=0\ (k\neq 0)$.

The level curves of f *are the equation line* $y=-x$ *for* $k=0$ *and the equation curves* $1+x^2+y^2-\dfrac{1}{k}x-\dfrac{1}{k}y=0$.

$$1+\left(x^2-2\frac{1}{2k}x\right)+\left(y^2-2\frac{1}{2k}y\right)=(x-\frac{1}{2k})^2+(x-\frac{1}{2k})^2+1-\frac{1}{2k^2}$$

These are circles with center $(\frac{1}{2k}, \frac{1}{2k})$ *and radius* $r = \sqrt{\frac{1}{2k^2} - 1}$ *with* $0 < k^2 \leq 1/2$*.*



2)$\partial_x f(x,y) = \dfrac{(1+x^2+y^2)-(x+y)2x}{(1+x^2+y^2)^2} = \dfrac{1-x^2-2xy+y^2}{(1+x^2+y^2)^2}$,

and $\partial_y f(x,y) = \dfrac{(1+x^2+y^2)-(x+y)2y}{(1+x^2+y^2)^2} = \dfrac{1+x^2-2xy-y^2}{(1+x^2+y^2)^2}$.

3) the equation of the tangent plane to f in (0, 0) is

$z = f(0,0) + x\,\partial_x f(0,0) + y\,\partial_y f(0,0)$ that is $z = x + y$.

Definition: (linearization)

Let be $f : \mathbb{R}^2 \to \mathbb{R}$ *a function defined on an* ***open set*** $D \subset \mathbb{R}^2$
and ***differentiable*** *at* $(x_0, y_0) \in D$.

We ***can approach the function*** f*, in the neighborhood of* (x_0, y_0)*,* ***by an affine function****:*

$$L(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\;(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\;(y-y_0).$$

The function $(h,k) \to E(h,k) = f(x_0+h, y_0+k) - L(x_0+h, y_0+k)$
measures the error *we make at the point* (x_0+h, y_0+k) *when we approach the value of* f *by the value of* L*; and since* f *is differentiable in* (x_0, y_0)*. then* $\lim_{h,k\to 0} \dfrac{E(h,k)}{\sqrt{h^2+k^2}} = 0$.

Example

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x\,e^{x\,y}$.

1) Show that f is differentiable and give its differential.

2) Give a linear approximation (of order 1) of $f(x,y)$ in the neighborhood of (1,0). Deduce an approximate value of $f(1.1,-0.1)$

Correction

1) f is composed of differentiable functions so it is differentiable.

We have $\partial_x f(x,y) = e^{x\,y} + xy\,e^{x\,y}$ and $\partial_y f(x,y) = x^2\,e^{x\,y}$ hence the differential:

$$df(x,y) = \partial_x f(x,y)\ dx + \partial_y f(x,y)\ dy = (e^{x\,y} + xy\,e^{x\,y})\ dx + (x^2\,e^{x\,y})\ dy.$$

2) Linearization f (***linear or order 1 approximation***) in the neighborhood of $(1,0)$ is

$$f(1+h,0+k) \approx df(1,0)(h,k) = f(1,0) + \partial_x f(1,0)\ h + \partial_y f(1,0)\ k.$$

Thus $f(1+h,0+k) \approx 1+h+k$; i.e. in the neighborhood of $(1,0)$ we have

$$x\,e^{x\,y} \approx x+y.$$

We deduce $f(1.1,-0.1) = f(1+0.1,-0.1) = 1+0.1-0.1 = 1$.
With a calculator we can see that $f(1.1,-0.1) = 0.985...$

Exercise

Knowing that a function $f:\mathbb{R}^2 \to \mathbb{R}$ is differentiable and that $f(2,5)=6$, $\partial_x f(2,5) = 1$, $\partial_y f(2,5) = -1$, give an approximate value of $f(2.2,4.9)$.

Correction

The function being differentiable, we can give an order 1 estimate of f.

In the neighborhood *of* $(x_0,y_0) = (2,5)$ *we have*

$$f(x,y) \approx f(x_0,y_0) + \partial_x f(2,5)\,(x-x_0) + \partial_y f(2,5)\,(y-y_0) = 6 + (x-2) - (y-5).$$

Especially $$f(2.2,4.9) \approx 6 + (0.2) - (-0.1) = 6.3.$$

Exercise

We measure a rectangle and we obtain a width of 30cm and a length of 24cm, with an error of at most 0.1cm for each measurement. Estimate the area of the rectangle.

Correction

The area of the rectangle is given by the function $f(x,y) = x\,y$ *with* x *the width and* y *the length (in* cm*). The function is differentiable so by* linearization *we can give an estimate of order 1 of the area* $f(x,y)$.

For (h,k) *small enough we have*

$$f(x+h, y+k) \approx f(x,y) + \partial_x f(x,y)\,h + \partial_y f(x,y)\,k = x \times y + y\,h + x\,k.$$

therefore , for $h,k \in [-0.1, 0.1]$, we have (noticing that f is increasing for each argument)

$$f(30+0.1, 24+0.1) \approx 30 \times 24 + 24\,0.1 + 30\,0.1 = 725.4,$$

$$f(30-0.1, 24-0.1) \approx 30 \times 24 + 24\,(-0.1) + 30\,(-0.1) = 714.6.$$

The area (denoted A) of the rectangle is between $714.6\,cm^2$ *and* $725.4\,cm^2$.

4 Multiple integrals

4.1. Functions with two real variables

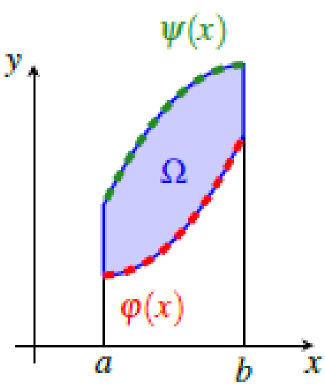
4.1.1. Fubini's theorem

We now present the integral of a function of two variables, called a double integral, and we show how to evaluate it.

Let φ *and* ψ *be two continuous functions on [a, b] with* $\varphi \leq \psi$.

Denote Ω *the set of points* $(x,y) \in \mathbb{R}^2$ *such that* $a \leq x \leq b$ *et* $\varphi(x) \leq y \leq \psi(x)$ *, then*

$$\iint_\Omega f(x,y)\,dx\,dy = \int_a^b \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\,dx.$$

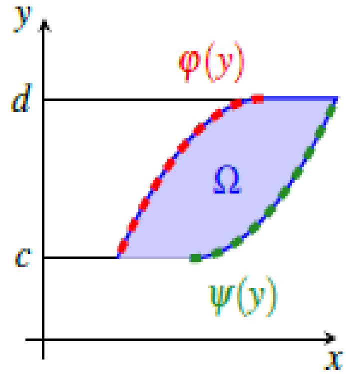


Fubini's theorem: If the domain allows it, we can swap the roles of x and y:

Let φ and ψ be two continuous functions on [c, d] with $\varphi \ \leq \ \psi$.

Denote Ω the set of points $(x, y) \in \mathbb{R}^2$ such that
$\varphi(x) \leq x \leq \psi(x)$ *et* $c \leq y \leq d$ *, then*

$$\iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \int_{\varphi(y)}^{\psi(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$



Example

Let be $\Omega = [0;1] \times [0;2]$; we want to calculate the double integral

$$\iint_\Omega xe^{xy}\ dx\,dy.$$

We have

$$\iint_\Omega xe^{xy}\ dx\,dy = \int_0^1 \int_0^2 xe^{xy}\ dy\,dx = \int_0^1 x\left(\int_0^2 e^{xy}\,dy\right)dx = \int_0^1 x\left[\frac{e^{xy}}{x}\right]_{y=0}^{y=2} dx$$
$$= \int_0^1 x\left[\frac{e^{2x}}{x} - \frac{1}{x}\right]dx = \int_0^1 \left[e^{2x} - 1\right]dx = \left[\frac{e^{2x}}{2} - x\right]_{x=0}^{x=1}$$
$$= \frac{e^2}{2} - 1 - \frac{1}{2} + 0 = \frac{e^2}{2} - \frac{3}{2}$$

Notice :

Sometimes, by using the Fubini's theorem, reversing the order of integration over an elementary domain, a double integral that is difficult to evaluate becomes relatively easy to solve.

Example

We want to calculate the volume of the solid which rises on the domain Ω of the plane *Oxy* delimited by the equation line $y = 2x$ and the parabola $y = x^2$a nd covered by the paraboloid $z = x^2 + y^2$.

The domain Ω is therefore delimited:

vertically by the paraboloid $z = x^2 + y^2$and the plane $z = 0$;
laterally by the line $y = 2x$ and the parabola $y = x^2$ which meets at $(x, y) = (0, 0)$ and $(x, y) = (2, 4)$ (*it suffices to solve* $y = 2x = x^2$).

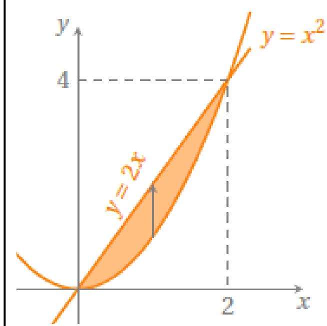
Then, the volume is given by $$V = \iint_\Omega x^2 + y^2 \; dx\, dy.$$

First method:

Le domaine Ω peut être décrit par

$$\Omega = \{(x, y) \in \mathbb{R}^2 \;:\; 0 \leq x \leq 2 \;,\; x^2 \leq y \leq 2x\}$$

$$\begin{aligned}
V = \iint_\Omega x^2 + y^2 \; dx\, dy &= \int_0^2 \Big(\int_{x^2}^{2x} (x^2 + y^2)\, dy \Big)\, dx = \int_0^1 \left[x^2\, y + \frac{y^3}{3} \right]_{y=x^2}^{y=2x} dx \\
&= \int_0^2 \left[x^2\,(2x) + \frac{(2x)^3}{3} - x^2\,(x^2) + \frac{(x^2)^3}{3} \right] dx \\
&= \int_0^2 \left[\frac{x^6}{3} - x^4 + \frac{14\, x^3}{3} \right] dx = \left[\frac{x^7}{21} - \frac{x^5}{5} + \frac{14\, x^4}{12} \right]_{x=0}^{x=2} \\
&= \frac{216}{35}
\end{aligned}$$

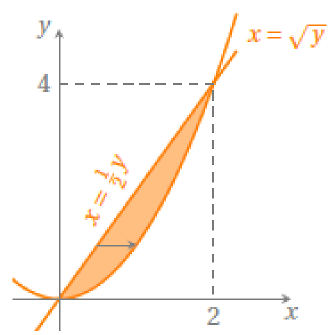


Second method.

Le domaine Ω peut être décrit par

$$\Omega = \{(x, y) \in \mathbb{R}^2 \;:\; 0 \leq y \leq 4 \;,\; \sqrt{y} \leq x \leq \frac{1}{2} y\}$$

$$\begin{aligned}
V = \iint_\Omega x^2 + y^2 \; dx\, dy &= \int_0^4 \Big(\int_{\sqrt{y}}^{\frac{y}{2}} (x^2 + y^2)\, dx \Big)\, dy = \int_0^4 \left[\frac{x^3}{3} + y^2\, x \right]_{x=\sqrt{y}}^{x=\frac{y}{2}} dx \\
&= \int_0^2 \left[\frac{y^3}{24} + \frac{y^3}{2} - \frac{(\sqrt{y})^3}{3} + y^2\,(\sqrt{y}) \right] dy \\
&= \int_0^2 \left[\frac{13y^3}{24} - \frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} \right] dy = \left[\frac{13y^4}{96} - \frac{2\, y^{\frac{5}{2}}}{15} + \frac{2\, y^{\frac{7}{2}}}{7} \right]_{y=0}^{y=4} \\
&= \frac{216}{35}
\end{aligned}$$



4.1.2. Special case (separable variables)

If Ω is the rectangle $[a, b] \times [c, d]$ and if f can be written in the form
$$f(x, y) = g(x)\; h(y)\,,$$
then

$$\int_a^b \int_c^d f(x, y)\, dx\, dy = \int_a^b g(x)\, dx \times \int_c^d h(y)\, dy$$

Example:

Let $\Omega = [0,1] \times [0,2]$, we want to calculate the double integral $\iint_\Omega x\, y\ dx\, dy$.

We have

$$\iint_\Omega dx\, dy = \int_0^1 \int_0^2 x\, y\, dx\, dy = \int_0^1 x\, dx \times \int_0^2 y\, dy$$
$$= \frac{x^2}{2}\Big|_0^1 \times \frac{y^2}{2}\Big|_0^2 = \frac{1}{2}$$

4.2. Applications

Definition: (Area)

*Let a **set** $D \subset \mathbb{R}^2$. The area of D is given by the integral*

$$\iint_D 1\ dx\, dy.$$

***Example** (Area of a disk)*

Let us calculate the area of a disk D_R of radius $R > 0$, we place ourselves in a coordinate system centered on the center of the disk, which therefore has the equation $x^2 + y^2 \le R^2$. So

$$D_R = \{(x,y) \in R^2 :\ x^2 + y^2 \le R^2\} = \{(r,\theta) \in R_+^* \times [0,2\pi[:\ r \le R\}$$

$$\iint_{D_R} 1\ dx\, dy = \int_0^{2\pi} \int_0^R r\ dr\, d\theta = \int_0^{2\pi} \frac{R^2}{2}\ d\theta = \pi\, R^2$$

Définition 35 (Volume) L'intégrale triple $\iiint_V 1\, dx\, dy\, dz$ mesure le volume de V.

Definition: (Volume)

*Let a **set** $V \subset \mathbb{R}^3$. The volume of V is given by the integral*

$$\iiint_V 1\ dx\, dy\, dz.$$

***Example** (Volume of a sphere)*

Let us calculate the area of a volume of a ball B_R of radius $R > 0$ (without restricting the generality we will assume it centered at the origin)

$$D_R = \{(x, y, z) \in R^2 : \; x^2 + \; y^2 \; + \; z^2 \leq \; R^2\,\}$$
$$= \;\left\{(r, \theta, \varphi) \in R_+^* \times [0{,}2\pi[\times [-{}^{\pi}/_{2}, {}^{\pi}/_{2}[: \; r \; \leq \; R\,\right\}$$

$$\begin{aligned} \iiint_{B_R} 1 \; dx\, dy\, dz \; &= \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^R r^2 \; \cos(\phi) \; dr\, d\theta\, d\phi \\ &= \left(\int_{-\pi/2}^{\pi/2} \cos(\phi)\, d\phi \right) \; \left(\int_0^{2\pi} 1\, d\theta \right) \; \left(\int_0^R r^2\, dr \right) \\ &= 2 \times 2\pi \times \frac{R^3}{3} = \frac{4}{3}\pi R^3. \end{aligned}$$

Algebra Part

Chap 5 : Matrices.

1 DEFINITIONS

Definition:

- *A* ***matrix*** *A is a* ***rectangular array*** *of elements of $K=\mathbb{R}$ or $K=\mathbb{C}$.*
- *It is said to be of* ***dimension m × n*** *if the table has* ***m rows*** *and* ***n columns****.*
- *The numbers in the table are called* ***coefficients*** *of A.*
- *The* ***coefficient located in*** *the* ***i-th line (****line number i****)*** *and in the* ***j-th column (****column number j****)*** *is noted $\boldsymbol{a_{i,j}}$.*
- *The* ***zero matrix****, denoted $O_{m,n}$, is the matrix whose all* ***elements are zero****.*
- *Two* ***matrices are equal*** *when they have the* ***same size*** *and* ***equal corresponding coefficients.***
- *The* ***set of matrices*** *with n rows and p columns with coefficients in K is denoted* ***$M_{m,n}(K)$.***

Notation: we will denote

$$\mathbb{A}=\begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix} \quad \text{or} \quad \mathbb{A}=\begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Or more simply $\mathbb{A}=(a_{ij})_{\substack{1\le i\le m\\1\le j\le n}}$ or $\mathbb{A}=[a_{ij}]_{\substack{1\le i\le m\\1\le j\le n}}$.

We can find $\mathbb{A}=(a_{ij})$ or $\mathbb{A}=[a_{ij}]$ if there is no confusions about dimension.

Examples.

1) Matrices of dimension 2 × 3. $A=\begin{pmatrix} 1 & -2 & 5 \\ 0 & 3 & 7 \end{pmatrix}$. $a_{1.1}=1$ and $a_{2.3}=7$.

2) Square matrix of order 3. $\mathbb{A}=\begin{pmatrix} -1 & 4 & 2 \\ 0 & 1 & -3 \\ 4 & 1 & 5 \end{pmatrix}$, $a_{2,2}=a_{3,2}=1$.

Particular matrices: *Here are some interesting matrix types*

- ✓ *If $m = n$ (same number of rows as columns), the matrix is called a* ***square matrix****. We* ***note*** ***Mn(K)*** *instead of Mn,n(K).*
  The elements ***$a_{1,1}$, $a_{2,2}$,..., $a_{n,n}$*** *form the main* ***diagonal*** *of the matrix.*
- ✓ *A matrix that has* ***only one row*** *(m = 1) is called* ***row matrix*** *or* ***row vector****.* ***We notice it*** $A = (a_{1,1},\ a_{1,1},\ \dots a_{1,n})$.
- ✓ Similarly, a matrix that has only one column (n = 1) is called a ***column matrix*** or a ***column vector***.

$$A = \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{n,1} \end{pmatrix}$$

- ✓ We call ***diagonal matrix*** any ***square matrix*** **D = $(d_{ij})_{1\le i,j\le n}$** such that ***$d_{ij} = 0$*** for all $i \neq j$. We notice it **Diag(d_1, d_2, . . . , d_n)**

$$\mathbb{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- ✓ *The matrix of order n, denoted* ***I_n****, is the* ***diagonal matrix*** **Diag*(1,1,...,1).***

$$\mathbb{I}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Kronecker symbol :. If i and j are two integers, we call Kronecker symbol, is the real number ***δ_{ij}***, which is 0 if i is different from j, and 1 if i is equal to j.

$$\delta_{i,j} = \begin{cases} 0 & \text{si } \ i \neq j \\ 1 & \text{si } \ i = j. \end{cases}$$

Then the ***general term of the identity matrix I is δ_{ij}*** .

- ✓ *We say that a square matrix $A = (a_{ij})_{1\le i,j\le n}$ is*
  - ❖ ***upper triangular*** *: if :* ***$i > j \Rightarrow a_{ij} = 0$,***
  - ❖ ***lower triangular*** *: if :* ***$i < j \Rightarrow a_{ij} = 0$.***
  - ❖ *An* ***upper and lower triangular*** *matrix is a* ***diagonal*** *matrix.*

$$\mathbb{U} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -5 \end{pmatrix}$$

$$\mathbb{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 5 & -1 & 2 & 0 \\ 7 & 9 & 15 & 4 \end{pmatrix}$$

$$\mathbb{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 5 & -1 & 2 & 0 \\ 7 & 9 & 15 & 4 \end{pmatrix}$$

2 OPERATIONS ON MATRICES

2.1. Sum and products

Addition of matrices:

If A and B are two matrices with the ***same size m × n*** *their* ***sum***
C = A+ B *is a matrix of the same size* ***m × n*** *defined by*

$$c_{ij} = a_{ij} + b_{ij}$$

Product of matrices with scalars:

If ***A =(aij)*** *is a* ***matrix*** *and* ***α*** *is a* ***scalar****, then their* ***product*** *is defined by*

$$\alpha A = (\alpha \times a_{ij}).$$

Properties:

Let A, B and C be matrices of same dimensions. Let α and β be two scalars.

1. A+ B = B + A: ***the sum is commutative****,*

2. A+ (B + C) = (A+ B) + C: ***the sum is associative****,*

3. A+ 0 = A: the ***null matrix is the neutral element*** *of the addition,*

4. (α + β)A = αA+ βA : ***matrices distribute upon scalars****,*

5. α(A+ B) = αA+ αB : ***scalars distribute upon matrices****.*

Examples.

1) For $\mathbb{A}=\begin{pmatrix}3&4&2\\1&3&5\end{pmatrix}$ and $\mathbb{B}=\begin{pmatrix}6&1&9\\2&0&3\end{pmatrix}$ we have $\mathbb{A}+\mathbb{B}=\begin{pmatrix}3+6&4+1&2+9\\1+2&3+0&5+3\end{pmatrix}=\begin{pmatrix}9&5&11\\3&3&8\end{pmatrix}$

2) For $A=\begin{pmatrix}3&-2\\1&7\end{pmatrix}$ and $B=\begin{pmatrix}0&5\\2&-1\end{pmatrix}$ we have $A+B=\begin{pmatrix}3&3\\3&6\end{pmatrix}$.

3) If $A=\begin{pmatrix}3&-2\\1&7\end{pmatrix}$ and $B'=\begin{pmatrix}-2\\8\end{pmatrix}$ then $A+B'$ doesn't exist.

The sum of two matrices of **different orders** is **not defined**.

4) For $\mathbb{A}=\begin{pmatrix}1&-1\\3&0\end{pmatrix}$, $\mathbb{B}=\begin{pmatrix}6&-5\\2&1\end{pmatrix}$, $\mathbb{C}=\begin{pmatrix}0&2\\2&4\end{pmatrix}$ then

$\mathbb{A}+\mathbb{B}=\begin{pmatrix}1+6&-1-5\\3+2&0+1\end{pmatrix}=\begin{pmatrix}7&-6\\5&1\end{pmatrix}$, $\mathbb{B}+\mathbb{A}=\begin{pmatrix}6+1&-5-1\\2+3&1+0\end{pmatrix}=\begin{pmatrix}7&-6\\5&1\end{pmatrix}$, $\mathbb{B}+\mathbb{C}=\begin{pmatrix}6+0&-5+2\\2+2&1+4\end{pmatrix}=\begin{pmatrix}6&-3\\4&5\end{pmatrix}$.

We can verify that $(\mathbb{A}+\mathbb{B})+\mathbb{C}=\begin{pmatrix}7 & -4\\ 7 & 5\end{pmatrix}$ and $\mathbb{A}+(\mathbb{B}+\mathbb{C})=\begin{pmatrix}7 & -4\\ 7 & 5\end{pmatrix}$.

5) If $\mathbb{A}=\begin{pmatrix}3 & 4 & 2\\ 1 & 3 & 5\end{pmatrix}$ and $\alpha=\frac{1}{2}$ then their product is $\alpha\cdot\mathbb{A}=\begin{pmatrix}3/2 & 2 & 1\\ 1/2 & 3/2 & 5/2\end{pmatrix}$.

Exercise.

Let $\mathbb{A}=\begin{pmatrix}-3 & 2\\ 0 & 4\\ 1 & -1\end{pmatrix}$ and $\mathbb{B}=\begin{pmatrix}1 & 2\\ 0 & 1\\ 1 & 1\end{pmatrix}$ be matrices.

1. Find matrix C such that A - 2B - C = O.

2. Find matrix D such that A + B + C - 4D = O

Correction

1. A - 2B - C = O is equivalent to C = A - 2B, *i.e*

$$\begin{pmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\\ c_{31} & c_{32}\end{pmatrix}=\begin{pmatrix}-3 & 2\\ 0 & 4\\ 1 & -1\end{pmatrix}-2\begin{pmatrix}1 & 2\\ 0 & 1\\ 1 & 1\end{pmatrix}=\begin{pmatrix}-3-2\times1 & 2-2\times2\\ 0-2\times0 & 4-2\times1\\ 1-2\times1 & -1-2\times1\end{pmatrix}=\begin{pmatrix}-5 & -2\\ 0 & 2\\ -1 & -3\end{pmatrix}$$

2. A + B + C - 4D = O is equivalent to D = ¼(A + B + C) , replacing C = A - 2B we get D = ¼ A + ½ B

$$\begin{pmatrix}d_{11} & d_{12}\\ d_{21} & d_{22}\\ d_{31} & d_{32}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-3 & 2\\ 0 & 4\\ 1 & -1\end{pmatrix}-\frac{1}{4}\begin{pmatrix}1 & 2\\ 0 & 1\\ 1 & 1\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\times(-3)-\frac{1}{4}\times1 & \frac{1}{2}\times2-\frac{1}{4}\times2\\ \frac{1}{2}\times0-\frac{1}{4}\times0 & \frac{1}{2}\times4-\frac{1}{4}\times1\\ \frac{1}{2}\times1-\frac{1}{4}\times1 & \frac{1}{2}\times(-1)-\frac{1}{4}\times1\end{pmatrix}=\begin{pmatrix}-7/4 & 1/2\\ 0 & 7/4\\ 1/4 & -3/4\end{pmatrix}$$

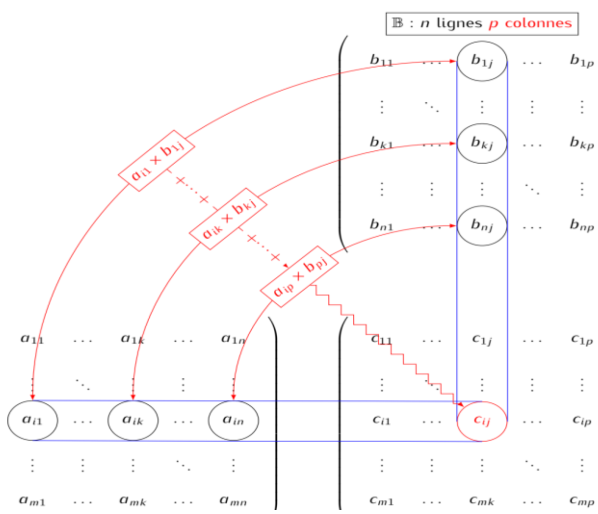
Product of matrices:

If $\mathbb{A}=(a_{ik})$ is an $\boldsymbol{m\times n}$ matrix and $\mathbb{B}=(b_{kj})$ an $\boldsymbol{n\times p}$ matrix, their product is defined by

$$\mathbb{A}\times\mathbb{B}=\left(\sum_{k=1}^{n}a_{ik}b_{kj}\right)_{\substack{1\le i\le m\\ 1\le j\le p}},$$

That, is A.B=C with

$$c_{ij}=\sum_{k=1}^{p}a_{ik}b_{kj}$$



Properties:

Let A, B and C be ***matrices of same dimensions****. Let α and β be scalars.*

1. *A(BC) = (AB)C :* *the product is associative*,

2. *A (B + C) = AB + AC and (B + C) A = BA+ CA :*
the product distribute upon addition,

3. *A·× 0 = 0 and 0 ×A = 0 : ,*

3. *Let A be a* ***m*** × ***n*** *matrix, then (I is neutral for product of matrix)*

$$I_m \times A = A \qquad \text{and} \qquad A \times I_n = A.$$

Examples.

1) Consider $A=\begin{pmatrix}1&2&3\\2&3&4\end{pmatrix}$ of ***size*** 2 × 3 and $B=\begin{pmatrix}1&2\\-1&1\\1&1\end{pmatrix}$ of ***size*** 3 × 2. The product is possible, it is a matrix of ***size*** 2 × 2. To calculates the first coefficient

$$C_{11} = 1\times 1 + 2\times(-1) + 3\times 1 = 2$$

(***sum of products of elements of the*** *lign*$_1$ ***and*** *column*$_1$)

We continue for the coefficient c_{12} with (***sum of products of elements of the*** *lign*$_1$ ***and*** *column*$_2$)

$$C_{12} = 1\times 2 + 2\times 1 + 3\times 1 = 7.$$

$$\begin{array}{cc} & \begin{pmatrix}1&2\\-1&1\\1&1\end{pmatrix}\\ \begin{pmatrix}1&2&3\\2&3&4\end{pmatrix} & \begin{pmatrix}c_{11}&c_{12}\\c_{21}&c_{22}\end{pmatrix}\end{array}$$

$$\begin{array}{cc} & \begin{pmatrix}1&2\\-1&1\\1&1\end{pmatrix}\\ \begin{pmatrix}1&2&3\\2&3&4\end{pmatrix} & \begin{pmatrix}2&c_{12}\\c_{21}&c_{22}\end{pmatrix}\end{array}$$

2) For $\mathbb{A}=\begin{pmatrix}1&3&0\\-1&1&2\end{pmatrix}$ and $\mathbb{B}=\begin{pmatrix}1&2&0\\0&2&3\\0&-1&-2\end{pmatrix}$ we have

$$\mathbb{A}\times\mathbb{B}=\begin{pmatrix}1\times 1+3\times 0+0\times 0 & 1\times 2+3\times 2+0\times(-1) & 1\times 0+3\times 3+0\times(-2)\\ -1\times 1+1\times 0+2\times 0 & -1\times 2+1\times 2+2\times(-1) & -1\times 0+1\times 3+2\times(-2)\end{pmatrix}=\begin{pmatrix}1&7&9\\-1&-2&-1\end{pmatrix}.$$

3) An interesting case is the **product** of a **row** vector by a column **vector**:

For $u=\begin{pmatrix}a_1 & a_2 & \cdots & a_n\end{pmatrix}$ and $v=\begin{pmatrix}b_1\\b_2\\\vdots\\b_n\end{pmatrix}$, then the product is a number (scalar):

$$\boxed{u\times v = a_1b_1 + a_2b_2 + \cdots + a_nb_n}$$

It is the ***scalar product*** of the vectors u and v .

Attention

1) The ***product of matrices is not commutative in general***.

$$\begin{pmatrix}5 & 1\\ 3 & -2\end{pmatrix}\begin{pmatrix}2 & 0\\ 4 & 3\end{pmatrix}=\begin{pmatrix}14 & 3\\ -2 & -6\end{pmatrix} \quad \text{is different of} \quad \begin{pmatrix}2 & 0\\ 4 & 3\end{pmatrix}\begin{pmatrix}5 & 1\\ 3 & -2\end{pmatrix}=\begin{pmatrix}10 & 2\\ 29 & -2\end{pmatrix}.$$

2) $AB = 0 \not\Rightarrow A = 0 \ or\ B = 0$

$$A=\begin{pmatrix}0 & -1\\ 0 & 5\end{pmatrix} \quad B=\begin{pmatrix}2 & -3\\ 0 & 0\end{pmatrix} \quad \text{and} \quad AB=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

***3)* Consequently** $AB = AC \not\Rightarrow A = 0 \ or\ B = C$.

$$A=\begin{pmatrix}0 & -1\\ 0 & 3\end{pmatrix} \quad B=\begin{pmatrix}4 & -1\\ 5 & 4\end{pmatrix} \quad C=\begin{pmatrix}2 & 5\\ 5 & 4\end{pmatrix} \text{and} \quad AB=AC=\begin{pmatrix}-5 & -4\\ 15 & 12\end{pmatrix}.$$

Exercise.

Compute the following operations

1) $\begin{pmatrix}3 & 1 & 5\\ 2 & 7 & 0\end{pmatrix}\times\begin{pmatrix}2 & 1 & -1 & 0\\ 3 & 0 & 1 & 8\\ 0 & -5 & 3 & 4\end{pmatrix}$ **2)** $\begin{pmatrix}-3 & 0 & 5\end{pmatrix}\times\begin{pmatrix}2\\ -4\\ -3\end{pmatrix}$ **3)** $\begin{pmatrix}2\\ -4\\ -3\end{pmatrix}\times\begin{pmatrix}-3 & 0 & 5\end{pmatrix}$

Correction

1) $\overbrace{\begin{pmatrix}3 & 1 & 5\\ 2 & 7 & 0\end{pmatrix}}^{2\times3}\times\overbrace{\begin{pmatrix}2 & 1 & -1 & 0\\ 3 & 0 & 1 & 8\\ 0 & -5 & 3 & 4\end{pmatrix}}^{3\times4}$

$$=\overbrace{\begin{pmatrix}3\times2+1\times3+5\times0 & 3\times1+1\times0+5\times(-5) & 3\times(-1)+1\times1+5\times3 & 3\times0+1\times8+5\times4\\ 2\times2+7\times3+0\times0 & 2\times1+7\times0+0\times(-5) & 2\times(-1)+7\times1+0\times3 & 2\times0+7\times8+0\times4\end{pmatrix}}^{2\times4}$$

$$=\begin{pmatrix}9 & -22 & 13 & 28\\ 25 & 2 & 5 & 56\end{pmatrix}$$

2) $\overbrace{\begin{pmatrix}-3 & 0 & 5\end{pmatrix}}^{1\times3}\times\overbrace{\begin{pmatrix}2\\ -4\\ -3\end{pmatrix}}^{3\times1}=\overbrace{\begin{pmatrix}-3\times2+0\times(-4)+5\times(-3)\end{pmatrix}}^{1\times1}=-21$

3) $\overbrace{\begin{pmatrix}2\\ -4\\ -3\end{pmatrix}}^{3\times1}\times\overbrace{\begin{pmatrix}-3 & 0 & 5\end{pmatrix}}^{1\times3}=\overbrace{\begin{pmatrix}2\times(-3) & 2\times0 & 2\times5\\ -4\times(-3) & -4\times0 & -4\times5\\ -3\times(-3) & -3\times0 & -3\times5\end{pmatrix}}^{3\times3}=\begin{pmatrix}-6 & 0 & 10\\ 12 & 0 & -20\\ 9 & 0 & -15\end{pmatrix}$

Powers of a matrix:

*In the set of **square matrices** $M_n(K)$, the **multiplication** of the matrices is an **internal operation** i.e. :*

*if **A, B** are in $M_n(K)$ then the product **A×B** is in $M_n(K)$.*

*We can then **repeat the multiplication**: $A2 = A\times A$, $A3 = A\times A\times A$.*

Thus, we can define the ***successive powers***

$$A^0 := I_n \qquad \text{and} \qquad A^{p+1} = A^p \times A \qquad \text{for all } p \geq 2.$$

That is : $\boxed{A^p = A \times A \times \cdots \times A}$ *for p factors.*

Example

We seek to calculate A^p for $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ where p is an integer number.

We calculate A2, A3 and A4 and we obtain:

$$A^2 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad A^3 = A^2 \times A = \begin{pmatrix} 1 & 0 & 7 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{pmatrix} \qquad A^4 = A^3 \times A = \begin{pmatrix} 1 & 0 & 15 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{pmatrix}.$$

The observation of these first powers makes it possible to think that the formula is:

$$A^p = \begin{pmatrix} 1 & 0 & 2^p - 1 \\ 0 & (-1)^p & 0 \\ 0 & 0 & 2^p \end{pmatrix}$$

Let us <u>*prove this result by induction*</u>: It is true for p = 0 (**$A^p = I_n$**).

We assume that it is **true for** an integer **p** and we will **prove it for p+1**. We have, by the definition

$$A^{p+1} = A^p \times A = \begin{pmatrix} 1 & 0 & 2^p - 1 \\ 0 & (-1)^p & 0 \\ 0 & 0 & 2^p \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2^{p+1} - 1 \\ 0 & (-1)^{p+1} & 0 \\ 0 & 0 & 2^{p+1} \end{pmatrix}$$

Which affirm that for all $p \geq 2$ $\quad A^p = \begin{pmatrix} 1 & 0 & 2^p - 1 \\ 0 & (-1)^p & 0 \\ 0 & 0 & 2^p \end{pmatrix}.$

2.2. Particular operations on matrices

Here are some interesting and useful operations on matrices.

✓ <u>***Matrix transpose***</u>

If A = (a_{ij}) is an **m × n** matrix, we define the ***transpose matrix*** of A, denoted A^T, by

$$A^T = (a_{ji}).$$

$$\mathbb{A} = \begin{pmatrix} 1 & -1 & 5 \\ 3 & 0 & 7 \end{pmatrix}$$

$$\mathbb{A}^T = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 5 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & -6 \\ -7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & -7 \\ 2 & 5 & 8 \\ 3 & -6 & 9 \end{pmatrix}$$

Properties

$(A^T)^T = A$, if $A \in M_{m,n}(K)$,

$(\alpha A)^T = \alpha A^T$ if $\alpha \in K$ and $A \in M_{m,n}(K)$,

$(A + B)^T = A^T + B^T$, if $A, B \in M_{m,n}(K)$,

An important case:

$$(1 \quad -2 \quad 5)^T = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

✓ ***Symmetric matrix***

The matrix A is said to be ***symmetric*** if

$$A^T = A$$

i.e. if $a_{ij} = a_{ji}$ for all $i \neq j$.

$$\mathbb{A} = \begin{pmatrix} 1 & 5 & -9 \\ 5 & 4 & 0 \\ -9 & 0 & 7 \end{pmatrix}$$

Symmetric matrix:

✓ ***Anti-symmetric matrix***

The matrix A is said to be ***antisymmetric*** if

$$A^T = - A$$

i.e. if $a_{ij} = - a_{ji}$ for all $i \neq j$.

Symmetric matrix:

$$\mathbb{B} = \begin{pmatrix} 1 & 5 & -9 \\ -5 & 4 & 0 \\ 9 & 0 & 7 \end{pmatrix}$$

✓ ***Trace of a matrix***

the ***trace*** **of a square matrix** A of order n, is the **sum of the elements of the diagonal** main.

$$\operatorname{tr}(\mathbb{A}) \equiv \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}.$$

Properties

Let **A** and **B** be **squares** matrices $(n \times n)$, then

1. **tr(A+ B) = trA + tr B,**
2. **tr(αA) = α trA** for all scalars α in K,

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$$

$$\operatorname{tr} A = 2 + 5 = 7$$

$$B = \begin{pmatrix} 1 & 1 & 2 \\ 5 & 2 & 8 \\ 11 & 0 & -10 \end{pmatrix}$$

$$\operatorname{tr} B = 1 + 2 - 10 = -7$$

Exercices.

Find x value such that the trace of matrix A is minimal.

Find x value such that the trace of matrix A is maximal.

$$\mathbb{A} = \begin{pmatrix} 2x^3 & 4 & 1 \\ 0 & 3x^2 & 2 \\ 5 & 6 & -12x \end{pmatrix}$$

Correction

Let's consider the function $y : x \to y(x) = tr(A)$; we have

$$y(x) = 2x3 + 3x2 - 12x.$$

$y'(x) = 6(x2 + x - 2)$, $y>0$ for $x < -2$ and $x > 1$, $y<0$ for $-2 < x < 1$.

Then tr(A) is maximal for $x = -2$ and tr(A) is minimal for $x = 1$.

- ✓ ***Invertible matrix***

A **square matrix A** ∈ Mn(K) is *said* to be ***invertible*** (or ***regular***) if there **exists** a matrix **B** ∈ Mn(K) such that

$$\mathbf{A \times B = B \times A = I_n}.$$

In this case, we note it $\mathbf{B = A^{-1}}$, it is **unique** and it is called ***inverse matrix*** of A.

- ✓ ***singular matrix***

A **non-invertible** matrix is said to be ***singular***.

Properties

Let **A** and **B** be two **invertible** matrices, then

- $\mathbf{A^{-1}}$ is also **invertible** and $\mathbf{(A^{-1})^{-1} = A}$,
- **A×B** is also **invertible** and $\mathbf{(A\times B)^{-1}=B^{-1}\times A^{-1}}$,
- $\mathbf{A^T}$ is also **invertible** and $\mathbf{(A^T)^{-1} = (A^{-1})^T}$.

Examples:

1) Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, we are looking for its inverse $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

If it exists, we must have

$AB = I$, then $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $\begin{pmatrix} a+2c & b+2d \\ 3c & 3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which deal to

$$\begin{cases} a+2c = 1 \\ b+2d = 0 \\ 3c = 0 \\ 3d = 1 \end{cases}$$

Solutions of this system is : $a = 1,\ b = -\frac{2}{3},\ \ c = 0,\ d = \frac{1}{3}$. The inverse is

$$A^{-1} = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}.$$

2) The identity **In** is **invertible**, and its inverse is itself by the equality:

In × **In** = **In**.

3) If **C** is **invertible**, one **can simplify** the equality $AC = BC$. Indeed:

multiplying $AC = BC$ ***on the right*** by C^{-1} yields to $(AC)C^{-1} = (BC)C^{-1}$.

By associativity we'll get $A(CC^{-1}) = B(CC^{-1})$.

This deals to $AI = BI$ which gives $A = B$.

Example **(singular matrices)** ;

1) Consider $A = \begin{pmatrix} 3 & 0 \\ 5 & 0 \end{pmatrix}$. If it exists, it invers $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ must verifies

$$BA = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 3a+5b & 0 \\ 3c+5d & 0 \end{pmatrix}.$$

The **product never can be equal to identity**. So **A** is a **singular** matrix.

2) The zero matrix **0n** of size **n × n** is not invertible because for any matrix B we have **B** x **0n** = **0n**, which can never be the identity matrix.

3 INVERSE OF A MATRIX CALCULUS

We are going to see a method to calculate the inverse of any matrix in an efficient way.

3.1. Square 2x2 matrices

We start with a simple formula for the elementary case of 2x2 matrices.

Proposition

Consider the 2x2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *If* $ad-bc\neq 0$, then is also **invertible** and

$$A^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Examples:

We **reconsider** the matrix of **example** ***1)*** above: $A=\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, we apply directly the given formula

$$A^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}=\frac{1}{3}\begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & -2/3 \\ 0 & 1/3 \end{pmatrix}.$$

3.2. Gaussian method for inverting matrices

We will take a look at an efficient approach for finding the inverse of any matrix. It is a linear system reformulation of the Gaussian pivot method.

Method

To invert a matrix **A**, first we write the ***augmented matrix*** **(A | I)** then we **perform** ***elementary operations*** on the rows of (A | I) **until** the table **(I | B)** is **obtained**. We conclude that $\mathbf{B = A^{-1}}$.

✓ ***Basic row operations:*** ***(to do in both sides of* (A | I))**

1. **Li ← λLi** (λ ≠ 0): **multiply** a line by a **non-zero scalar**.

2. **Li ← Li** + λLj (j ≠ i): **add** to the line Li a multiple of another line Lj.

3. **Li ← Lj** : we can **exchange** two lines

✓ ***Equivalent matrices***

Two **matrices** are said to be ***equivalent*** if one **resuts** from the other **by elementary operations**.

Examples:

Let be the matrix $A=\begin{pmatrix} 1 & 2 & 1 \\ 4 & 0 & -1 \\ -1 & 2 & 2 \end{pmatrix}$. We consider the augmented matrix

$$(A\,|\,I) = \begin{pmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 4 & 0 & -1 & | & 0 & 1 & 0 \\ -1 & 2 & 2 & | & 0 & 0 & 1 \end{pmatrix} \begin{matrix} L_1 \\ L_2 \\ L_3 \end{matrix}.$$

We apply **elementary operations** to make **zeros below the diagonal**.
We will ***obtain*** a ***lower triangular matrix***.

To make 0 appear on the **first column**, apply **L2 ← L2 - 4L1** for the second line and **L3 ← L3 + L1** for the third line:

$$\begin{pmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 0 & -8 & -5 & | & -4 & 1 & 0 \\ -1 & 2 & 2 & | & 0 & 0 & 1 \end{pmatrix}_{L_2 \leftarrow L_2 - 4L_1} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 0 & -8 & -5 & | & -4 & 1 & 0 \\ 0 & 4 & 3 & | & 1 & 0 & 1 \end{pmatrix}_{L_3 \leftarrow L_3 + L_1}.$$

Multiply the line L2 to get 1 in the diagonal, apply **L2 ← L2 - 4L1** :

$$\begin{pmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{8} & | & \frac{1}{2} & -\frac{1}{8} & 0 \\ 0 & 4 & 3 & | & 1 & 0 & 1 \end{pmatrix} L_2 \leftarrow -\tfrac{1}{8}L_2.$$

We repeat the procedure for the **second column**:

$$\begin{pmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{8} & | & \frac{1}{2} & -\frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} & | & -1 & \frac{1}{2} & 1 \end{pmatrix}_{L_3 \leftarrow L_3 - 4L_2}, \quad \begin{pmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{8} & | & \frac{1}{2} & -\frac{1}{8} & 0 \\ 0 & 0 & 1 & | & -2 & 1 & 2 \end{pmatrix}_{L_3 \leftarrow 2L_3}$$

We'll do the same to make appear **zeros above the diagonal:**
We will ***obtain*** a ***upper triangular matrix***.

$$\begin{pmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & \frac{7}{4} & -\frac{3}{4} & -\frac{5}{4} \\ 0 & 0 & 1 & | & -2 & 1 & 2 \end{pmatrix} L_2 \leftarrow L_2 - \tfrac{5}{8}L_3, \quad \begin{pmatrix} 1 & 0 & 0 & | & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & | & \frac{7}{4} & -\frac{3}{4} & -\frac{5}{4} \\ 0 & 0 & 1 & | & -2 & 1 & 2 \end{pmatrix}^{L_1 \leftarrow L_1 - 2L_2 - L_3}.$$

Hence the inverse matrix of A is $A^{-1} = \frac{1}{4}\begin{pmatrix} -2 & 2 & 2 \\ 7 & -3 & -5 \\ -8 & 4 & 8 \end{pmatrix}$.

4 DETERMINANTS

4.1. Definition and practical computation

Definitions

A being a **square matrix** of **order n**. For all **(i,j)**, **1 ≤ i, j ≤ n**, we denote by **A_{ij}** the **square matrix** of **order n – 1** obtained by **deleting** the **i-th row** and the **j-th column** of A.

$$A=\begin{pmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & & \vdots & \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i,1} & \cdots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \cdots & a_{i,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \vdots & & \vdots & & \vdots & \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j} & a_{n,j+1} & \cdots & a_{nn} \end{pmatrix} \qquad A_{ij}=\begin{pmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & & & \vdots \\ a_{n,1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n} \end{pmatrix}$$

The number $(-1)^{i+j}\det A_{ij}$ is called ***cofactor*** associated to the element **a_{ij}**.

The matrix whose elements are cofactors $(-1)^{i+j}\det A_{ij}$ is called the ***co-matrix*** and is denoted ***Com(A)***.

Calculus determinants methods

The ***determinant* of A**, denoted **det(A)** or **|A|**, is defined **by induction**:

- **if n = 1**: **det(A)** ≡ **a_{11}**
- **if n ≥ 2**: **distributing co-matrices** upon their corresponding elements **a_{ij}** we'll get :

1) $\det(\mathbb{A}) \equiv \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det(\mathbb{A}_{ij})$ **distributing along the line i;**

$$=(-1)^{i+1}a_{i,1}\det A_{i,1}+(-1)^{i+2}a_{i,2}\det A_{i,2}+\cdots+(-1)^{i+n}a_{i,n}\det A_{i,n}.$$

or

2) $\det(\mathbb{A}) \equiv \sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det(\mathbb{A}_{ij})$ **distributing along the column j.**

Applications: **(**n=2 and **n=3)**

1) case n = 2.: Let the matrix $\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then

det(**A**11) = a22, det(**A**12) = a21, det(**A**21) = a12, det(**A**22) = a11.

We can **calculate det(A)** by one of the following formulas (***as examples***):

- ✓ **development along line i = 1 :**

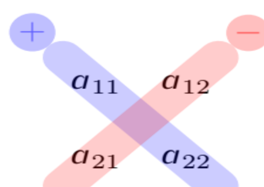
$$a_{11}det(A_{11}) - a_{12}det(A_{12}) = a_{11}a_{22} - a_{12}a_{21}\ ,$$

- ✓ **development along column j = 1 :**

$$a_{11}det(A_{11}) - a_{21}det(A_{21}) = a_{11}a_{22} - a_{21}a_{21}\ ,$$

which give the same result.

$$\det(\mathbb{A}) = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}\ ;$$



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Example: $\det\begin{pmatrix} 5 & 7 \\ 4 & 3 \end{pmatrix} = 5 \times 3 - 7 \times 4 = 15 - 28 = -13.$

2) case n = 3.: Let the matrix $\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then choosing to

develop along the first line (**i=1**), i.e. the elements $a_{11}\quad a_{12}\quad a_{13}$, the determinants of the corresponding co-matrices are

$$\det(\mathbb{A}_{11}) = \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} = a_{22}a_{33} - a_{23}a_{32},\ \det(\mathbb{A}_{12}) = \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} = a_{21}a_{33} - a_{23}a_{31},$$

$$\det(\mathbb{A}_{13}) = \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = a_{21}a_{32} - a_{22}a_{31}.$$

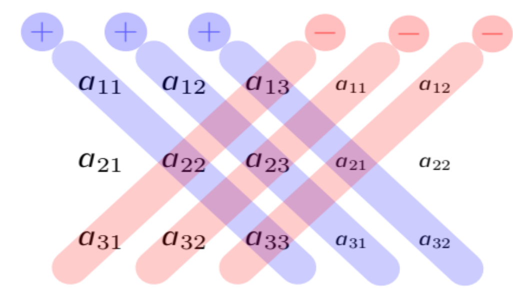
Then $\det(\mathbb{A}) = a_{11}\det(\mathbb{A}_{11}) - a_{12}\det(\mathbb{A}_{12}) + a_{13}\det(\mathbb{A}_{13})$.

It's much easier to remember

$$\det(\mathbb{A}) = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

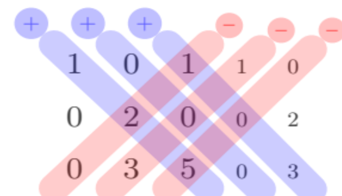
$$= \left(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{33} + a_{13}a_{21}a_{32}\right)$$

$$- \left(a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}\right)$$



(Sarrus rule's)

Example: $\mathbb{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 3 & 5 \end{pmatrix}$



$$\det(\mathbb{A}) = (1\times 2\times 5 + 0\times 0\times 0 + 1\times 0\times 3) - (1\times 2\times 0 + 1\times 0\times 3 + 0\times 0\times 5) = 10.$$

Remark:

For **n ≥ 4**, we have to choose a line or a column which contain many zeros and **distribute along this line** [**0** x **det(Aij) =0**].

Application:

1) Let be the matrix $\mathbb{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 4 \\ 1 & 2 & 3 & 0 \end{pmatrix}$. We **distribute along the line** 1.

$$\det(\mathbb{A}) = \det(\mathbb{A}_{11}) - \det(\mathbb{A}_{14}) = \det\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 4 \\ 2 & 3 & 0 \end{pmatrix} - \det\begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix}.$$

For the first matrix we **distribute along the column** 2, for the second matrix we will use the **rule of Sarrus.**

$$= -\det\begin{pmatrix} 2 & 4 \\ 2 & 0 \end{pmatrix} - \left(12+0+2-2-0-0\right) = -(-8) - 12 = -4.$$

2) Let be the matrix $A=\begin{pmatrix}4&0&3&1\\4&2&1&0\\0&3&1&-1\\1&0&2&3\end{pmatrix}$. We **distribute along the column** 2.

$$\det A = +2\begin{vmatrix}4&3&1\\0&1&-1\\1&2&3\end{vmatrix}-3\begin{vmatrix}4&3&1\\4&1&0\\1&2&3\end{vmatrix}$$

$$= +2\left(+4\begin{vmatrix}1&-1\\2&3\end{vmatrix}-0\begin{vmatrix}3&1\\2&3\end{vmatrix}+1\begin{vmatrix}3&1\\1&-1\end{vmatrix}\right)-3\left(-4\begin{vmatrix}3&1\\2&3\end{vmatrix}+1\begin{vmatrix}4&1\\1&3\end{vmatrix}-0\begin{vmatrix}4&3\\1&2\end{vmatrix}\right)$$

$$= +2\big(+4\times5-0+1\times(-4)\big)-3\big(-4\times7+1\times11-0\big) = 83$$

Exercise.

Calculate the determinant of the following matrices

$$\mathbb{A}=\begin{pmatrix}1&3\\-7&5\end{pmatrix},\qquad \mathbb{B}=\begin{pmatrix}2&3&3\\5&15&7\\0&-2&0\end{pmatrix},\qquad \mathbb{C}=\begin{pmatrix}2&-1&3&-4\\2&0&4&-5\\-2&4&3&1\\0&-3&1&-1\end{pmatrix}.$$

Correction

1. $\det(\mathbb{A}) = 1\times5-3\times(-7)=26.$

2. Let's develop along the last line :

$\det(\mathbb{B}) = (-1)^{3+2}\times(-2)\times\det\begin{pmatrix}2&3\\5&7\end{pmatrix} = 2\times(2\times7-3\times5)=-2.$

3. *We will use elementary transforms to make appear zeros before developing along columns (this will simplify calculus)*

$$\mathbb{C}=\begin{pmatrix}2&-1&3&-4\\2&0&4&-5\\-2&4&3&1\\0&-3&1&-1\end{pmatrix}\xrightarrow[=]{\substack{L_2\leftarrow L_2-L_1\\L_3\leftarrow L_3+L_1}}\begin{pmatrix}2&-1&3&-4\\0&1&1&-1\\0&3&6&-3\\0&-3&1&-1\end{pmatrix}$$

then

$$\det(\mathbb{C})=\det\begin{pmatrix}2&-1&3&-4\\2&0&4&-5\\-2&4&3&1\\0&-3&1&-1\end{pmatrix}=2\times\det\begin{pmatrix}1&1&-1\\3&6&-3\\-3&1&-1\end{pmatrix}$$

Again

$$\begin{pmatrix}1&1&-1\\3&6&-3\\-3&1&-1\end{pmatrix}\xrightarrow[=]{\substack{L_2\leftarrow L_2-3L_1\\L_3\leftarrow L_3+3L_1}}\begin{pmatrix}1&1&-1\\0&3&0\\0&4&-4\end{pmatrix}$$

then

$$2 \times \det \begin{pmatrix} 1 & 1 & -1 \\ 3 & 6 & -3 \\ -3 & 1 & -1 \end{pmatrix} = 2 \times \left((-1)^{1+1} \times 1 \times \det \begin{pmatrix} 3 & 0 \\ 4 & -4 \end{pmatrix} \right)$$
$$= 2 \times (3 \times (-4) - 0 \times 4) = -24.$$

Exercise.

Calculate the determinant of the following matrices

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 2 & 3 & 7 & 4 \\ 3 & 1 & 12 & 0 \\ 4 & 0 & -5 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 3 & 4 \\ 1 & 7 & 12 & -5 \\ 0 & 3 & 1 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}$$

Correction

1. $\det \begin{pmatrix} 0 & 0 & \boxed{1} & 0 \\ 2 & 3 & 7 & 4 \\ 3 & 1 & 12 & 0 \\ 4 & 0 & -5 & 0 \end{pmatrix} = \det \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 0 \\ \boxed{4} & 0 & 0 \end{pmatrix} = 4 \det \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix} = -16.$

2. $\det \begin{pmatrix} 0 & 2 & 3 & 4 \\ \boxed{1} & 7 & 12 & -5 \\ 0 & 3 & 1 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix} = -\det \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 0 \\ \boxed{4} & 0 & 0 \end{pmatrix} = -4 \det \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix} = 16.$

4.2. Properties related to determinants

✓ ***Properties f determinant***

Let **A** and **B** be matrices, then

- **det(A^T) = det(A),**
- **det(A^{-1}) = 1/det(A),**
- **det(A × B) = det(A) × det(B),**
- if **B** is equivalent to **A** (***obtained via elementary transforms***) then **det(A) = det(B).**

✓ ***Reversibility***

A matrix **A** is **invertible** if and only if **det(A) ≠ 0**.

✓ ***Determinants of triangular matrix***

The determinant of an **upper** (or **lower**) **triangular matrix** is equal to the **product of the diagonal terms.**

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \cdots a_{nn}.$$

✓ ***Determinants of diagonal matrix***

Identic result: The determinant of a **diagonal matrix** is equal to the **product of the diagonal terms**.

Exercise.

Find t values such that the following matrices will be invertible

$$\mathbb{A} = \begin{pmatrix} t+3 & t^2-9 \\ t^2+9 & t-3 \end{pmatrix}$$

Correction

Recall that matrix A is invertible if and only if det(A)≠0.

$$\det(\mathbb{A}) = \det \begin{pmatrix} t+3 & t^2-9 \\ t^2+9 & t-3 \end{pmatrix} = (t+3) \times (t-3) - (t^2-9) \times (t^2+9) = -(t-3)(t+3)(t^2+8).$$

So, the matrix is invertible far all $t \in \mathbb{R} \setminus \{ -3, 3 \}$.

4.3. Rank of a matrix

Definition and results

✓ ***Definition***

The rank of an **m**x**n** matrix A, denoted **rg(A)**, is equal to the **largest integer s** such that one **can extract from A** an **invertible square matrix** of **order s** (*i.e. a nonzero determinant **square matrix** of **order s***).

✓ ***Results***

Always **$0 \leq rg(A) \leq \min(m, n)$**

$rg(A) = 0$ if and only if all the **elements of A are zero**.

Examples:

1) Let be the matrix $\mathbb{A} = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 1 \end{pmatrix}$.

- **Dimension** of **A** is **2 × 3** then **s ≤ min{2, 3}**, so s = 0, 1 or 2;
- at least one **element** of **A** is **different from zero**, so s ≠ 0;
- since the **determinant** of the **sub-matrix** composed of the first and the third column is **non-zero**, then s = 2.

2) Let be the matrix $\mathbb{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 5 & -1 \\ -1 & 0 & -1 \end{pmatrix}$.

- **Order** of **A** is **3 × 3** then **s ≤ 3**;
- at least one **element** of **A** is **different from zero**, so **s ≠ 0**;
- **det(A)=0** then **s ≠ 3**;
- the **determinant** of the **sub-matrix** $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ is **non-zero**, then **s = 2**.

Exercise.

Calculate the ranks of the following matrices

$$\mathbb{B} = \begin{pmatrix} 1 & 2 & 8 \\ 2 & 1 & 4 \\ 0 & 3 & 12 \end{pmatrix} \qquad \mathbb{C} = \begin{pmatrix} 2 & 1 & 3 \\ 8 & 4 & 12 \\ 1 & 2 & 0 \end{pmatrix}$$

Correction

1. $1 \leq rg(\mathbb{B}) \leq 3$. $\det(\mathbb{B}) = 0$ then $1 \leq rg(\mathbb{B}) \leq 2$.

$$\det \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = -3 \neq 0 \text{ then } rg(\mathbb{B}) = 2.$$

2. $1 \leq rg(\mathbb{C}) \leq 3$. $\det(\mathbb{C}) = 0$ then $1 \leq rg(\mathbb{C}) \leq 2$.

$$\det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3 \neq 0, \text{ then } rg(\mathbb{C}) = 2.$$

4.4. Inverse of a matrix

✓ ***Theorem***

Let A be an invertible matrix, and C its comatrix. Then we have

$$A^{-1} = \frac{1}{\det A} C^T$$

Examples:

Let be the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. We have det(A)=2, then **A** is invertible.

Calculating the **cofactors** of all elements of matrix A we'll get the **co-matrix**

$$C = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

We deduce

$$A^{-1} = \frac{1}{\det A} \cdot C^T = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Exercise.

Calculate, using two methods, the inverse matrix of

$$\mathbb{A} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Correction

1. We calculate the comatrix **Com(A)** (elements are cofactors)

$$\mathbf{Com(A)} = \begin{pmatrix} (-1)^{1+1}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{pmatrix};$$

Transpose comatrix : $\mathbf{Com(A)^T} = \begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{pmatrix};$

Dividing by **det(A)** we get : $$\mathbb{A}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

2. We will use **augmented matrix** and **elementariy transforms**

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_3 \leftarrow L_3 - L_1]{L_2 \leftarrow L_2 + L_1} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ \boxed{0} & 2 & 0 & 1 & 1 & 0 \\ \boxed{0} & -2 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{L_2 \leftarrow L_2/2} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 1/2 & 1/2 & 0 \\ 0 & -2 & 2 & -1 & 0 & 1 \end{array}\right) \xrightarrow[L_3 \leftarrow L_3 + 2L_2]{L_1 \leftarrow L_1 - L_2} \left(\begin{array}{ccc|ccc} 1 & \boxed{0} & -1 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & \boxed{0} & 2 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{L_3 \leftarrow L_3/2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & \boxed{1} & 0 & 1/2 & 1/2 \end{array}\right) \xrightarrow{L_1 \leftarrow L_1 + L_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \boxed{0} & 1/2 & 0 & 1/2 \\ 0 & 1 & \boxed{0} & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 \end{array}\right)$$

We deduce : $$\mathbb{A}^{-1} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

Chap 6 : System of linear equations

1 SYSTEM OF LINEAR EQUATIONS

Definition:

*$n, p \geq 1$ being integers, a **$n \times p$ linear system** is a set of n linear equations with p unknowns*

$$(S) \quad \begin{cases} a_{11}x_1 & + & \dots & + & a_{1p}x_p & = & b_1, \\ \vdots & & & & \vdots & & \vdots \\ a_{n1}x_1 & + & \dots & + & a_{np}x_p & = & b_n. \end{cases}$$

- ✓ *The **coefficients** a_{ij} and the **second members** b_i are given elements of $K = \mathbb{R}$ or $K = \mathbb{C}$. The **unknowns** $x_1, x_2, \dots, x_p$ are to be found in K.*
- ✓ *The **homogeneous system** associated with (S) is the system obtained by replacing $b_i=0$.*
- ✓ *A **solution** of (S) is a **p-tuple** $(x_1, x_2, \dots, x_p)$ which **satisfies simultaneously** the **n equations** of (S).*
- ✓ ***Solving** (S) is to search for all solutions.*
- ✓ *A system is **impossible**, or **incompatible**, if it does not admit a solution.*
- ✓ *Two systems are **equivalent** if they have the same solutions.*

Matrix writing:

If we denote

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad \mathbb{A} = \begin{pmatrix} a_{11} & \dots & a_{1p} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{np} \end{pmatrix}.$$

*Then, the **system** (S) is **equivalent** to the matrix writing **$Ax = b$**.*

$$(S)\quad \begin{cases} a_{11}x_1 + \dots + a_{1p}x_p = b_1, \\ \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{n1}x_1 + \dots + a_{np}x_p = b_n. \end{cases} \textit{equivalent} \begin{pmatrix} a_{11} & \dots & a_{1p} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{np} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Exercise.

Reconsider the matrix $\mathbb{A} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ of the precedent exercise.

1) Write the system equivalent to the matrix equation **AX=B**. Precise the nature of X and B.

2) Solve the equation **AX=B**.

Correction

1) The matrix equation **AX=B** is equivalent to the system

$$(S)\ \begin{cases} 1\,x_1 + 1\,x_2 - 1\,x_3 = b_1 \\ -1\,x_1 + 1\,x_2 + 1\,x_3 = b_2 \\ 1\,x_1 - 1\,x_2 + 1\,x_3 = b_3 \end{cases}$$

where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are vectors matrices.

2) To solve the equation **AX=B** we need matrix **A** to be invertible. It is and we have

$$\mathbb{A}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

Consequently, we have $\mathbf{X=A^{-1}B}$ and the inverse system of (S) is

$$(S^{-1})\ \begin{cases} 1/2\,b_1 + 0\,b_2 + 1/2\,b_3 = x_1 \\ 1/2\,b_1 + 1/2\,b_2 + 0\,b_3 = x_2 \\ 0\,x_1 + 1/2\,b_2 + 1/2\,b_3 = x_3 \end{cases}$$

2 CRAMER'S METHOD

Consider a system of ***n equations*** *and* ***n unknowns***

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{cases}$$

This system can be written in matrix form ***AX = B*** *where*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in M_n(\mathbb{K}), \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Cramer's rule:

Define the matrix $A_j \in M_n(\mathbb{K})$ *obtained by* ***replacing the j-th column of A by*** *the second member* ***B***

$$A_j = \begin{pmatrix} a_{11} & \dots & a_{1,j-1} & b_1 & a_{1,j+1} & \dots & a_{1n} \\ a_{21} & \dots & a_{2,j-1} & b_2 & a_{2,j+1} & \dots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{n,j-1} & b_n & a_{n,j+1} & \dots & a_{nn} \end{pmatrix}$$

The Cramer's rule says that: if ***det(A)≠0*** (***that is A is invertible***), then the unique solution $(x_1, x_2, \dots, x_n)$ of the system (***it is also to the matrix equation***) is given by

$$x_1 = \frac{\det A_1}{\det A} \qquad x_2 = \frac{\det A_2}{\det A} \qquad \dots \qquad x_n = \frac{\det A_n}{\det A}.$$

Exercise.

Apply for the system of the exercise below.

Correction

The system

$$(S)\ \begin{cases} 1\,x_1 + 1\,x_2 - 1\,x_3 = b_1 \\ -1\,x_1 + 1\,x_2 + 1\,x_3 = b_2 \\ 1\,x_1 - 1\,x_2 + 1\,x_3 = b_3 \end{cases}$$

is equivalent to the matrix equation

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

We have det(A)=1/4 (not zero) then the system (the matrix equation) admits a unique solution (x_1, x_2, x_3) given by

$$x_1 = \frac{\begin{vmatrix} b_1 & 1 & -1 \\ b_2 & 1 & 1 \\ b_3 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}} = \frac{b_1 \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} - b_2 \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + b_3 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}}{4} = \frac{2\,b_1 + 0\,b_2 + 2\,b_3}{4}$$

$$x_2 = \frac{\begin{vmatrix} 1 & b_1 & -1 \\ -1 & b_2 & 1 \\ 1 & b_3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}} = \frac{-b_1 \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} + b_2 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - b_3 \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}}{4} = \frac{2\,b_1 + 2\,b_2 + 0\,b_3}{4}$$

$$x_3 = \frac{\begin{vmatrix} 1 & 1 & b_1 \\ -1 & 1 & b_2 \\ 1 & -1 & b_3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}} = \frac{b_1 \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - b_2 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + b_3 \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}}{4} = \frac{0\,b_1 + 2\,b_2 + 2\,b_3}{4}.$$

Comparing with the system (S^{-1}) we notice the same results.

$$(S^{-1})\ \begin{cases} 1/2\,b_1 + 0\,b_2 + 1/2\,b_3 = x_1 \\ 1/2\,b_1 + 1/2\,b_2 + 0\,b_3 = x_2 \\ 0\,x_1 + 1/2\,b_2 + 1/2\,b_3 = x_3 \end{cases}$$

Exercise.

Resolve the system
$$\begin{cases} x_1 \phantom{{}+4x_2} + 2x_3 = 6 \\ -3x_1 + 4x_2 + 6x_3 = 30 \\ -x_1 - 2x_2 + 3x_3 = 8. \end{cases}$$

Correction

The is equivalent to the matrix equation **AX=B** where

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 6 \\ 30 \\ 8 \end{pmatrix}$$

Developing along the first line we get

$$det(A) = 1\begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix} - 0\begin{vmatrix} -3 & 6 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix} = 24 + 20 = 44$$

$$x_1 = \frac{\begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix}}{44} = \frac{6\begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix} - 0\begin{vmatrix} 30 & 6 \\ 8 & 3 \end{vmatrix} + 2\begin{vmatrix} 30 & 4 \\ 8 & -2 \end{vmatrix}}{44} = \frac{6\,(24) + 2\,(92)}{44} = \frac{-40}{44}$$

$$x_2 = \frac{\begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix}}{44} = \frac{1\begin{vmatrix} 30 & 6 \\ 8 & 3 \end{vmatrix} - 6\begin{vmatrix} -3 & 6 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} -3 & 30 \\ -1 & 8 \end{vmatrix}}{44} = \frac{72}{44}$$

$$x_3 = \frac{\begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix}}{44} = \frac{1\begin{vmatrix} 4 & 30 \\ -2 & 8 \end{vmatrix} - 0\begin{vmatrix} -3 & 30 \\ -1 & 8 \end{vmatrix} + 6\begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix}}{44} = \frac{152}{44}$$

3 GAUSSIAN PIVOT METHOD

Stepped system:

*A system **(S)** is **stepped**, or **Staggered**, if the number of first successive zero coefficients of each equation is strictly increasing.*
*The corresponding **matrix** is **triangular**.*

Example:

The following system is **stepped**.

$$\left\{\begin{array}{rrrrrrcl} 5x_1 & -x_2 & -x_3 & +2x_4 & & +x_6 & = & b_1 \\ & & 3x_3 & -x_4 & +2x_5 & & = & b_2 \\ & & & & -x_5 & +x_6 & = & b_3 \\ & & & & & 5x_6 & = & b_4 \\ & & & & & 0 & = & b_5 \end{array}\right.$$

> ***Method:***
> Gaussian pivot **method** consists to ***transform*** **a system to a stepped one** (*triangular matrix*), we will **use elementary operations** on lines of this system (*or on column of the augmented matrix*).
> **Solutions** will be deduced **successively because** of the **triangularization**.

Exercise.

Resolve the system
$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 1, \\ 2x_1 + 3x_2 + 4x_3 + x_4 = 2, \\ 3x_1 + 4x_2 + x_3 + 2x_4 = 3, \\ 4x_1 + x_2 + 2x_3 + 3x_4 = 4. \end{cases}$$

Correction

1. Resolution by the Gaussian pivot method:

we will use elementary transforms to reduce this system to a stepped one (*triangular matrix*)

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 1 \\ 2x_1 + 3x_2 + 4x_3 + x_4 = 2 \\ 3x_1 + 4x_2 + x_3 + 2x_4 = 3 \\ 4x_1 + x_2 + 2x_3 + 3x_4 = 4 \end{cases} \xrightarrow[\text{Étape 1}]{\substack{L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 3L_1 \\ L_4 \leftarrow L_4 - 4L_1}} \begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 1 \\ -x_2 - 2x_3 - 7x_4 = 0 \\ -2x_2 - 8x_3 - 10x_4 = 0 \\ -7x_2 - 10x_3 - 13x_4 = 0 \end{cases}$$

$$\xrightarrow[\text{Étape 2}]{\substack{L_3 \leftarrow L_3 - 2L_2 \\ L_4 \leftarrow L_4 - 7L_2}} \begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 1 \\ -x_2 - 2x_3 - 7x_4 = 0 \\ -4x_3 + 4x_4 = 0 \\ 4x_3 + 36x_4 = 0 \end{cases} \xrightarrow[\text{Étape 3}]{L_4 \leftarrow L_4 + L_3} \begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 1 \\ -x_2 - 2x_3 - 7x_4 = 0 \\ -4x_3 + 4x_4 = 0 \\ 40x_4 = 0 \end{cases}$$

We deduce successively

$$\begin{aligned} & x_4 = 0 \\ & -4\,x_3 + 4 \times 0 = 0 \implies x_3 = 0 \\ & -x_2 - 2 \times 0 - 7 \times 0 = 0 \implies x_2 = 0 \\ & x_1 + 2 \times 0 - 2 \times 0 - 7 \times 0 = 1 \implies x_1 = 1 \end{aligned}$$

2. Resolution by the Gaussian pivot method in matrix writing:

$$[\mathbb{A}|\mathbf{b}] = \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 3 & 4 & 1 & 2 \\ 3 & 4 & 1 & 2 & 3 \\ 4 & 1 & 2 & 3 & 4 \end{array}\right) \xrightarrow[\text{Étape 1}]{\substack{L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 3L_1 \\ L_4 \leftarrow L_4 - 4L_1}} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & -1 & -2 & -7 & 0 \\ 0 & -2 & -8 & -10 & 0 \\ 0 & -7 & -10 & -13 & 0 \end{array}\right)$$

$$\xrightarrow[\text{Étape 2}]{\substack{L_3 \leftarrow L_3 - 2L_2 \\ L_4 \leftarrow L_4 - 7L_2}} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & -1 & -2 & -7 & 0 \\ 0 & 0 & -4 & 4 & 0 \\ 0 & 0 & 4 & 36 & 0 \end{array}\right) \xrightarrow[\text{Étape 3}]{L_4 \leftarrow L_4 + L_3} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & -1 & -2 & -7 & 0 \\ 0 & 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 40 & 0 \end{array}\right)$$

We deduce successively

$$\begin{aligned} & 40\,x_4 = 0 \implies x_4 = 0 \\ & -4\,x_3 + 4 \times 0 = 0 \implies x_3 = 0 \\ & -x_2 - 2 \times 0 - 7 \times 0 = 0 \implies x_2 = 0 \\ & x_1 + 2 \times 0 - 2 \times 0 - 7 \times 0 = 1 \implies x_1 = 1 \end{aligned}$$

Exercise.

Solve the system
$$\begin{cases} -2u - 4v + 3w = -1 \\ \quad 2v - w = 1 \\ u + v - 3w = -6 \end{cases}$$

Correction

$$\begin{cases} -2u - 4v + 3w = -1 \\ \quad 2v - w = 1 \\ u + v - 3w = -6 \end{cases} \xrightarrow{L_3 \leftarrow L_3 + L_1/2} \begin{cases} -2u - 4v + 3w = -1 \\ \quad 2v - w = 1 \\ \quad -v - \frac{3}{2}w = -13/2 \end{cases} \xrightarrow{L_3 \leftarrow L_3 + L_2/2} \begin{cases} -2u - 4v + 3w = -1 \\ \quad 2v - w = 1 \\ \quad -2w = -6 \end{cases}$$

We deduce successively $w = 3$, $v = 2$ et $u = 1$.

Exercise.

Solve the system
$$\begin{pmatrix} 6 & 1 & 1 \\ 2 & 4 & 0 \\ 1 & 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 6 \end{pmatrix}$$

Correction

$$[\mathbb{A}|\mathbf{b}] = \left(\begin{array}{ccc|c} 6 & 1 & 1 & 12 \\ 2 & 4 & 0 & 0 \\ 1 & 2 & 6 & 6 \end{array}\right) \xrightarrow{\substack{L_2 \leftarrow L_2 - \frac{2}{6}L_1 \\ L_3 \leftarrow L_3 - \frac{1}{6}L_1}} \left(\begin{array}{ccc|c} 6 & 1 & 1 & 12 \\ 0 & \frac{11}{3} & -\frac{1}{3} & -4 \\ 0 & \frac{11}{6} & \frac{35}{6} & 4 \end{array}\right) \xrightarrow{L_3 \leftarrow L_3 - \frac{\frac{11}{6}}{\frac{11}{3}} L_2} \left(\begin{array}{ccc|c} 6 & 1 & 1 & 12 \\ 0 & \frac{11}{3} & -\frac{1}{3} & -4 \\ 0 & 0 & 6 & 6 \end{array}\right)$$

We deduce $\begin{cases} 6x_1 + x_2 + x_3 = 12, \\ \frac{11}{3}x_2 - \frac{1}{3}x_3 = -4 \\ 6x_3 = 6 \end{cases}$ then successively $x_3 = 1, \quad x_2 = -1, \quad x_1 = 2.$

Exercise.

Consider the linear system $(S) \quad \begin{cases} x + y - z = 1, \\ 2x + 3y + \beta z = 3, \\ x + \beta y + 3z = -3. \end{cases}$

Discuss according to the real parameter β, the solutions of the system (S).

Correction

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & \beta & 3 \\ 1 & \beta & 3 & -3 \end{array}\right) \xrightarrow[L_3 \leftarrow L_3 - L_1]{L_2 \leftarrow L_2 - 2L_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \beta+2 & 1 \\ 0 & \beta-1 & 4 & -4 \end{array}\right) \xrightarrow{L_3 \leftarrow L_3 + (1-\beta)L_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \beta+2 & 1 \\ 0 & 0 & (6-\beta-\beta^2) & -(3+\beta) \end{array}\right)$$

Notice that $6 - \beta - \beta 2 = (2 - \beta)(3 + \beta)$ we conclude

1. if $\beta = -3$ then the last equation is $0z = 0$: hens (S) has an infinity of solutions,

2. if $\beta = 2$ then the last equation is $0z = -5$ (impossible): hens (S) has no solution,

3. if $\beta \neq 2$ and $\beta \neq -3$ then (S) has a unique solution

$$\begin{aligned} &(2-\beta)(3+\beta)\,z = -(3+\beta) \implies z = \frac{-1}{2-\beta} \\ &y + (\beta+2)\,z = y - \frac{\beta+2}{2-\beta} = 1 \implies y = 1 + \frac{\beta+2}{2-\beta} \\ &x + y - z = x + 1 + \frac{\beta+2}{2-\beta} - \frac{-1}{2-\beta} = 1 \implies y = 1 - \frac{\beta+3}{2-\beta} \end{aligned}$$

Exercise.

Consider the linear system $(S_a) \quad \begin{cases} (1+a)x + y + z = 0, \\ x + (1+a)y + z = 0, \\ x + y + (1+a)z = 0, \end{cases}$

Discuss according to the real parameter a, the solutions of the system (S_a).

Correction

Let us check the determinant of (S_a)

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = (1+a)^3 + 1 + 1 - (1+a) - (1+a) - (1+a) = (1+a)^3 - 3(1+a) + 2$$

$$= a^2(3+a).$$

(Sa) is a Cramer system if and only a ∈ R \ { −3, 0 } . In this case (Sa) will have a unique solution.

Case a = −3 : the system yields to

$$(S_{-3}) \quad \begin{cases} -2x + y + z = 0, \\ x - 2y + z = 0, \\ x + y - 2z = 0, \end{cases}$$

Using the Gaussian pivot method, we'll get

$$\begin{cases} -2x \quad y \ +z=0 \\ x-2y \ +z=0 \\ x \quad y-2z=0 \end{cases} \xrightarrow[L_3 \leftarrow L_3 + L_1/2]{L_2 \leftarrow L_2 + L_1/2} \begin{cases} -2x \quad y \ +z=0 \\ -\frac{3}{2}y+\frac{3}{2}z=0 \\ \frac{3}{2}y-\frac{3}{2}z=0 \end{cases} \xrightarrow{L_3 \leftarrow L_3 + L_2} \begin{cases} -2x \ -y \ +z=0 \\ -\frac{5}{2}y+\frac{3}{2}z=0 \\ 0z=0 \end{cases}$$

The last equation is true for any value of z and solutions set is

$$\mathcal{S} = \{ (\kappa, \kappa, \kappa) \mid \kappa \in \mathbb{R} \}.$$

Case a = 0 : the system yields to

$$(S_0) \quad \begin{cases} x + y + z = 0, \\ x + y + z = 0, \\ x + y + z = 0, \end{cases}$$

Which is equivalent to $x + y + z = 0$, so $z = \kappa_1 \in \mathbb{R}$, $y = \kappa_2 \in \mathbb{R}$ et $x = -\kappa_1 - \kappa_2$, and solutions set is

$$\mathcal{S} = \left\{ (-\kappa_1 - \kappa_2, \kappa_2, \kappa_1) \;\middle|\; (\kappa_1, \kappa_2) \in \mathbb{R}^2 \right\}$$

Case a ∈ R \ { −3, 0 } since it is a linear system and second member is zero then the solutions set is

$$\mathscr{S} = \{ (0, 0, 0) \}.$$

Exercise.

Consider the linear system $(S_a) \quad \begin{cases} x + ay + (a-1)z = 0, \\ 3x + 2y + az = 3, \\ (a-1)x + ay + (a+1)z = a, \end{cases}$

Discuss according to the real parameter a, the solutions of the system (S_a).

Correction

Let us check the determinant of (S_a)

$$\begin{vmatrix} 1 & a & a-1 \\ 3 & 2 & a \\ a-1 & a & a+1 \end{vmatrix} = 2(a+1) + a^2(a-1) + 3a(a-1) - 2(a-1)^2 - a^2 - 3a(a+1) = a^2(a-4).$$

(Sa) is a Cramer system if and only a ∈ R \ {0 , 4} . In this case (Sa) will have a unique solution.

Case a = 0 : the system yields to

$$(S_0) \quad \begin{cases} x - z = 0, \\ 3x + 2y = 3, \\ -x + z = 0, \end{cases}$$ which is equivalent to the system $$\begin{matrix} x - z = 0, \\ 3x + 2y = 3, \end{matrix}$$

Two equations with 3 unknowns, then putting z as parameter we'll get

$z = \kappa \in \mathbb{R}$, $y = \frac{3-3\kappa}{2}$ et $x = \kappa$, and solutions set is

$$\mathcal{S} = \left\{ \left(\kappa, \frac{3-3\kappa}{2}, \kappa \right) \,\middle|\, \kappa \in \mathbb{R} \right\}.$$

Case a = 4 : the system yields to

$$(S_4) \quad \begin{cases} x + 4y + 3z = 0, \\ 3x + 2y + 4z = 3, \\ 3x + 4y + 5z = 4, \end{cases}$$

Using the Gaussian pivot method, we'll get

$$\begin{cases} x+4y+3z=0, \\ 3x+2y+4z=3, \\ 3x+4y+5z=4, \end{cases} \xrightarrow[L_3 \leftarrow L_3 - 3L_1]{L_2 \leftarrow L_2 - 3L_1} \begin{cases} x \ +4y+3z=0, \\ \ -10y-5z=3, \\ \ -8y-4z=4, \end{cases} \xrightarrow{L_3 \leftarrow 10L_3 - 8L_2} \begin{cases} x \ +4y+3z=0, \\ \ -10y-5z=3, \\ \ 0=16. \end{cases}$$

The last equation is impossible, then solutions set is $\mathcal{S} = \emptyset.$

Case a ∈ R \ { 0, 4 } system has a unique solutions. Let's use the Gaussian pivot method

$$\begin{cases} x+ay+(a-1)z=0, \\ 3x+2y \quad +az=3, \\ (a-1)x+ay+(a+1)z=a, \end{cases} \xrightarrow[L_3 \leftarrow L_3 - (a-1)L_1]{L_2 \leftarrow L_2 - 3L_1} \begin{cases} x \quad +ay \ +(a-1)z=0, \\ (2-3a)y+(3-2a)z=3, \\ (2-a)ay+(3-a)az=a, \end{cases}$$

$$\xrightarrow{L_3 \leftarrow L_3 - \frac{(2-a)a}{(2-3a)} L_2} \begin{cases} x \quad +ay \ +(a-1)z=0, \\ (2-3a)y+(3-2a)z=3, \\ -\frac{a^2(a-4)}{3a-2} z = \frac{4a}{3a-2}. \end{cases}$$

We get successively $z=-\frac{4}{a(a-4)}$, $y=-\frac{a-6}{a(a-4)}$, $x=\frac{a^2-2a-4}{a(a-4)}$, hens

$$\mathscr{S}=\left\{\left(\frac{a^2-2a-4}{a(a-4)},-\frac{a-6}{a(a-4)},-\frac{4}{a(a-4)}\right)\right\}.$$

4 GAUSS-JORDAN METHOD

> **Method:**
>
> *In this variation, the main is **make appear zeros both above and below the pivot**. In this case we end up with a **diagonal system**.*
>
> **Solutions** will be obtained **directly because** of the **diagonalization**

Exercise.

Solve the equation $$\begin{pmatrix}1&2&3&4\\2&3&4&1\\3&4&1&2\\4&1&2&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$$

Correction

$$[\mathbb{A}|\mathbf{b}]=\left(\begin{array}{cccc|c}1&2&3&4&1\\2&3&4&1&2\\3&4&1&2&3\\4&1&2&3&4\end{array}\right)$$

$$\xrightarrow[\text{Étape 1}]{\substack{L_2\leftarrow L_2-2L_1\\L_3\leftarrow L_3-3L_1\\L_4\leftarrow L_4-4L_1}}\left(\begin{array}{cccc|c}1&2&3&4&1\\0&-1&-2&-7&0\\0&-2&-8&-10&0\\0&-7&-10&-13&0\end{array}\right)\xrightarrow[\text{Étape 2}]{\substack{L_1\leftarrow L_1+2L_2\\L_3\leftarrow L_3-2L_2\\L_4\leftarrow L_4-7L_2}}\left(\begin{array}{cccc|c}1&0&-1&-10&1\\0&-1&-2&-7&0\\0&0&-4&4&0\\0&0&4&36&0\end{array}\right)$$

$$\xrightarrow[\text{Étape 3}]{\substack{L_1\leftarrow L_1-L_3/4\\L_2\leftarrow L_2-L_3/2\\L_4\leftarrow L_4+L_3}}\left(\begin{array}{cccc|c}1&0&0&4&1\\0&-1&0&-7&0\\0&0&-4&4&0\\0&0&0&40&0\end{array}\right)\xrightarrow[\text{Étape 4}]{\substack{L_1\leftarrow L_1+11L_4/40\\L_2\leftarrow L_2+9L_4/40\\L_3\leftarrow L_3+4L_4/40}}\left(\begin{array}{cccc|c}1&0&0&0&1\\0&-1&0&0&0\\0&0&-4&0&0\\0&0&0&40&0\end{array}\right)$$

We deduce successively

$1\,x_1=1\implies x_1=1;\ -x_2=0\implies x_2=0;\ -4\,x_3=0\implies x_3=0;\ 40\,x_4=0\implies x_4=0.$